Recall: • Laplace transform (LT)

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

• Inverse Laplace transform (ILT)

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$$ (table)

• Solve ODE:

(i) Take LT $(Y(s) = \mathcal{L}\{y(t)\}(s))$
??

(ii) Use properties of LT ($Y(s) = \ldots$ alg. eq.)

(iii) Take ILT $(y(t) = \mathcal{L}^{-1}\{Y\}(t))$

sol. to ODE :-)

What type of ODE can solve?

$$y''(t) - 2y(t) = \sin(t)$$
linear LT ok

$$y''(t) - 2\sin(y(t)) = 2$$ nonlinear
ok LT LT = ?? LT ok

→ Prob linear
→ Can compute all LT
⇒ OK!

5) Convolution:

$\rightsquigarrow$ .pdf

Chapter IX: Fourier analysis

Goal: Approximate functions by "simple" trigonometric functions ($\sin(x)$, $\cos(x)$)

Applications: Signal processing, .mp3, .jpeg

1) Periodic functions:

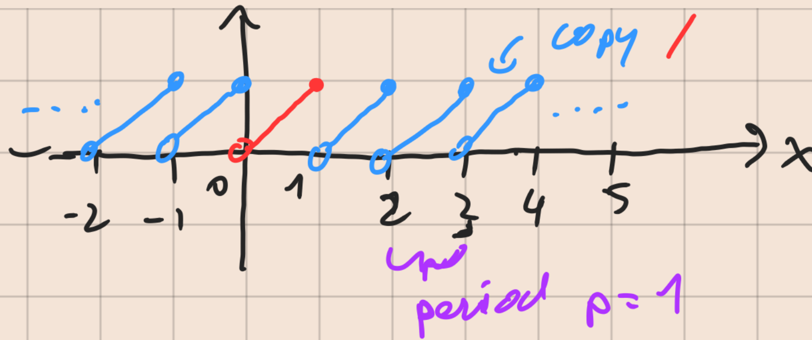
Def: • A function f is periodic if $\exists\, p>0$ s.t.

$$f(x+p)=f(x) \quad \forall x$$

• The smallest such p is called the (prime) period. In this case, f is called p-periodic.

Ex: • $\sin$ and $\cos$ are 2π-periodic

• $f(x)=x$ for $0<x\leq 1$ and extended periodically



Lemma: Let f be integrable and p-periodic.

Then, the integral

$$\int_a^{a+p} f(x)\,dx$$

does not depend on the point a!

Proof:

Assume $a<p$: $\displaystyle\int_a^{a+p} f(x)\,dx = \int_a^{p} f(x)\,dx + \int_p^{a+p} f(x)\,dx =$

$x = y+p$
$dx = dy$

$$= \int_a^{p} f(x)\,dx + \int_0^{a} \underbrace{f(y+p)}_{=f(y) \text{ since } f \text{ } p\text{-periodic}}\,dy = \underbrace{\int_0^{p} f(x)\,dx}_{\text{does not depend on } a!} \quad :-)$$

$x = p \overset{!}{=} y+p \Rightarrow y = 0$

$x = a+p \overset{!}{=} y+p \Rightarrow y = a$

$\blacksquare$

2) Fourier series:

Def: Let f be 2π-periodic and integrable on $[-\pi, \pi]$. The expression

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

is called the Fourier series of f (FS).

The coefficients a_n and b_n are called the Fourier coefficients of f are given

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \qquad \text{for } n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx \qquad \text{for } n = 1, 2, \dots$$

Rem: We can replace $\int_{-\pi}^{\pi}$ by $\int_{0}^{2\pi}$ or integrate on any intervals of length 2π (above Lemma)

In order to find a more compact form to FS, we use

$$e^{inx} = \cos(nx) + i\sin(nx)$$
$$e^{-inx} = \cos(nx) - i\sin(nx)$$
$$\Rightarrow \quad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \qquad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

Into FS, we get

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right)$$

$$= \frac{a_0}{2} + \sum_{n=0}^{\infty} \left\{ \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \frac{i}{i} \right) e^{-inx} \right\}$$

$$= \underbrace{\frac{a_0}{2}}_{=: c_0} + \sum_{n=0}^{\infty} \left\{ \underbrace{\left(\frac{a_n - ib_n}{2} \right)}_{=: c_n} e^{inx} + \underbrace{\left(\frac{a_n + ib_n}{2} \right)}_{=: c_{-n}} e^{-inx} \right\}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad // \qquad \text{hat: } \sum \xi_k \varphi_k(x)$$

↗

FS of f in a representation with complex numbers.

Rem: $c_0 = \frac{a_0}{2} \Leftrightarrow a_0 = 2c_0$

$$c_n = \frac{a_n - ib_n}{2} = \overline{c_{-n}} \Longleftrightarrow \begin{array}{l} a_n = c_n + c_{-n} \\ b_n = i(c_n - c_{-n}) \end{array}$$

complex conj.

3) Justification of the above formulas:

Lemma: The set $\{e^{inx}\}_{n=-\infty}^{\infty}$ is an orthogonal set on $[-\pi, \pi]$.

That is $$\int_{-\pi}^{\pi} e^{inx} \cdot e^{-ikx}\, dx = \begin{cases} 0 & \text{if } k \neq n \\ 2\pi & \text{if } k = n \end{cases}$$

Rem: Inner product in $[-\pi, \pi]$

$$(f,g) = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx$$

Proof:

$$\int_{-\pi}^{\pi} e^{inx} e^{-ikx}\,dx = \int_{-\pi}^{\pi} e^{i(n-k)x}\,dx =$$

$$= \begin{cases} n = k \to \int_{-\pi}^{\pi} e^{0}\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi \\[2ex] n \neq k \to \left.\dfrac{e^{i(n-k)x}}{i(n-k)}\right|_{x=-\pi}^{\pi} = \end{cases}$$

$$= \frac{e^{i(n-k)\pi} - e^{-i(n-k)\pi}}{i(n-k)} =$$

$$= \frac{\overset{(-1)^{n-k}}{\cancel{\cos((n-k)\pi)}} + \overset{=0}{\cancel{i\sin((n-k)\pi)}} - \left(\overset{(-1)^{n+k}}{\cancel{\cos((n-k)\pi)}} + \overset{0=}{\cancel{i\sin((n-k)x)}}\right)}{i(n-k)}$$

$$= 0$$

$\blacksquare$

The above can be used to justify the formulas for a_n, b_n, c_n:

Assume $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$,

Multiply by e^{-ikx} and $\int_{-\pi}^{\pi}$ to get:

$$\int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-ikx} dx}_{\substack{=0 \text{ if } n \neq k \\ =2\pi \text{ if } n=k}} =$$

$$= 2\pi c_k$$

$$\hookrightarrow c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \quad //$$

Once we know $c_k \rightsquigarrow a_0, a_k, b_k$:

$$a_0 \underset{\text{above}}{=} 2c_0 \underset{\text{Def } c_0}{=} \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)e^{0}\,dx =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad !\text{-})$$

Ex: Find the FS of $f(x) = x$ for $-\pi < x \leq \pi$ and extended periodically

- $$a_0 \underset{\text{Def } a_0}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \underset{\text{Def } f(x)}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx =$$

$$= \frac{1}{\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\pi^2 - (-\pi)^2\right) = 0$$

- $$a_n \underset{\text{Def } a_n}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x}_{v} \underbrace{\cos(nx)}_{u'} \, dx \overset{\text{by parts}}{=}$$

$\uparrow$

Def f