Piazza / Quiz

Recall: Polynomial interpolation

× data points
ex: $(x_j, f(x_j))$

Th: $\underbrace{\| \Pi_1 f - f \|}_{\text{error}}{}_{L^p(a,b)} \le C_1 \cdot (b-a)^2 \, \|f''\|_{L^p(a,b)}$

Proof:

$$|\Pi_1 f(x) - f(x)| \le \frac{1}{2} |x-x_0|^2 \, |f''(\xi_0)| \, |\lambda_0(x)| + \frac{1}{2} |x_1 - x|^2 \, |f''(\xi_1)| \, |\lambda_1(x)|$$

$$|\Pi_1 f(x) - f(x)| \le \frac{1}{2} \underbrace{|x-x_0|^2}_{\le |b-a|^2} \cdot \underbrace{|f''(\xi_0)|}_{\le \max\limits_{a \le x \le b} |f''(x)|} \cdot \underbrace{|\lambda_0(x)|}_{\le 1} + \frac{1}{2} \underbrace{|x_1 - x|^2}_{\le |b-a|^2} \cdot \underbrace{|f''(\xi_1)|}_{\le \max\limits_{a \le z \le b} |f''(z)|} \cdot \underbrace{|\lambda_1(x)|}_{\le 1}$$

since $a \le x \le b$

(Def Lagrange)

(λ_1, λ_0 ⇝ Lagrange poly)

$$\Rightarrow |\Pi_1 f(x) - f(x)| \le |b-a|^2 \cdot \underbrace{\max_{a \le z \le b} |f''(z)|}_{\|f''\|_{L^\infty(a,b)}} \quad \forall x \in [a,b]$$

(Def norm)

$$\Rightarrow \|\Pi_1 f - f\|_{L^\infty(a,b)} \le |b-a|^2 \cdot \|f''\|_{L^\infty(a,b)}$$

:-)

2) Continuous piecewise linear interpolation:

Goal: Replace polynomials with pw linear functions in the previous section

Recall: $V_h = \{ v : [a,b] \to \mathbb{R} : v \text{ is cont. pw linear on } T_h \}$,

where

T_h: uniform partition of $[a,b]$:

$$x_0 = a < x_1 < x_2 < \dots < x_{m+1} = b$$

with $x_j - x_{j-1} = h$

$V_h = \operatorname{span}(\varphi_0, \varphi_1, \dots, \varphi_{m+1})$ with φ_j hat fct.

We know that each $v \in V_h$ can be written as $v(x) = \sum_{j=0}^{m+1} \underbrace{\xi_j}_{\text{coordinates}} \cdot \underbrace{\varphi_j(x)}_{\text{basis}}$ (linear combination)

Def: Let $f : [a,b] \to \mathbb{R}$ continuous and a partition T_h as above. The continuous piecewise linear interpolant of f is given by

$$\Pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\cdot\varphi_j(x) \qquad \forall\, x\in[a,b]$$

$\Pi_h f \in V_h$

Ex: Consider $f(x)=(x-1)\cdot(x-2)+1$ for $x\in[1,3]$ and a uniform partition of $[1,3]$ into 2 subintervals. Find $\Pi_h f$.

We have the partition T_h: $\underset{1}{\underset{\shortparallel}{x_0}} < \underbrace{\underset{2}{\underset{\shortparallel}{x_1}} < \underset{3}{\underset{\shortparallel}{x_2}}}_{\text{"}h\text{"}}$, $x_2 = x_{m+1}$

Hence, $m=1$ and $h=\frac{b-a}{m+1}=\frac{3-1}{2}=1$

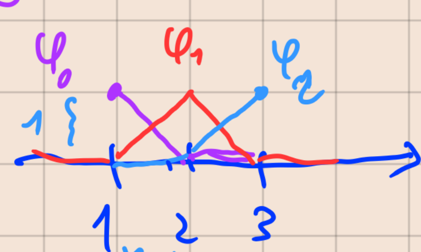
By definition, we have

$$\Pi_h f(x) = \sum_{j=0}^{2} f(x_j)\varphi_j(x) = f(x_0)\varphi_0(x)+f(x_1)\varphi_1(x) + f(x_2)\varphi_2(x)$$

But, we have $f(x_0)=f(1)=1$, $f(x_1)=f(2)=1$,

$$f(x_2)=f(3)=3$$

we also have



$h=1$

$\varphi_j(x_j)=1$

$\Rightarrow \Pi_h f(x) = 1\cdot\varphi_0(x)+1\cdot\varphi_1(x)+3\cdot\varphi_2(x) =$

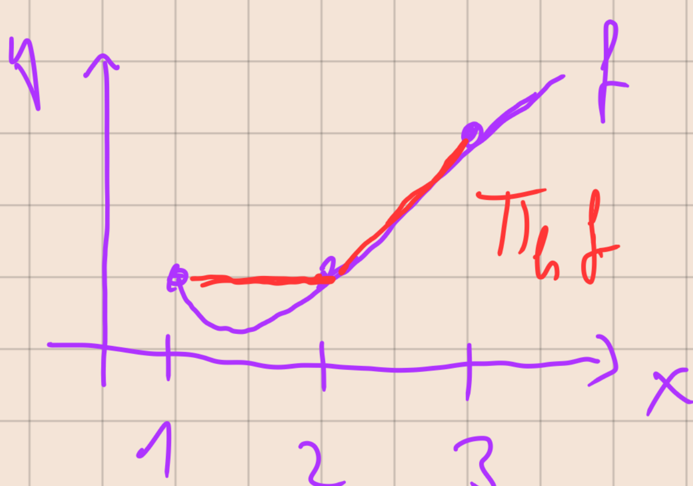
Def of φ_j $\rightarrow$

$$= \begin{cases} \frac{x-x_1}{-h}+\frac{x-x_0}{h}+0 = 1 & \text{if } 1\le x\le 2 \\ \cancel{0+\frac{x-x_2}{-h}+3\frac{x-x_1}{h}} = 2x-3 & \text{if } 2\le x\le 3 \end{cases}$$

$$\begin{cases} 0 + \dfrac{x-x_2}{-h} + 3\cdot\dfrac{x-x_1}{h} = 2x-3 \end{cases}$$

$$\Pi_h f(x) \underset{\text{Def } \varphi_0,\ \varphi_2}{=} \begin{cases} 1\cdot\varphi_0(x) + 1\cdot\varphi_1(x) + 3\cdot 0 & \text{if } 1 \le x \le 2 \\ 1\cdot 0 + 1\cdot\varphi_1(x) + 3\cdot\varphi_2(x) & \text{if } 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} 1\cdot\dfrac{x-x_1}{-h} + 1\cdot\dfrac{x-x_0}{h} & \text{if } 1 \le x \le 2 \\ 1\cdot\dfrac{x-x_2}{-h} + 3\cdot\dfrac{x-x_1}{h} & \text{if } 2 \le x \le 3 \end{cases}$$



if $m \uparrow$ or $h \downarrow$

$\rightarrow$ $\Pi_h f$ seems to approximate f better!

What is the error $\Pi_h f - f$??

$\rightarrow$ 2 times cont. differentiable

Th: Let $f \in C^2(a,b)$ and $\Pi_h f$ a cont. pw lin. interpolant. Then, $\exists$ constants C_1, C_2, C_3 s.t.

(i) $\|\Pi_h f - f\|_{L^p(a,b)} \leq C_1 \cdot h^{\boxed{2}} \cdot \|f''\|_{L^p(a,b)}$

(ii) $\|\Pi_h f - f\|_{L^p(a,b)} \leq C_2 \cdot h \cdot \|f'\|_{L^p(a,b)}$

(iii) $\|(\Pi_h f)' - f'\|_{L^p(a,b)} \leq C_3 \cdot h \cdot \|f''\|_{L^p(a,b)}$

for $p = 1, 2, \infty$.

Proof: (for (ii) for $p = 1, 2$)

$$\|\Pi_h f - f\|^p_{L^p(a,b)} \underset{\uparrow\ \text{Def norm}}{=} \int_a^b |f(x)|^p \, dx \underset{\uparrow\ a = x_0 < x_1 < x_2 \dots}{=}$$

$$= \sum_{j=0}^{m} \int_{x_j}^{x_{j+1}} |f(x)|^p \, dx = \sum_{j=1}$$

$$\|\Pi_h f - f\|^p_{L^p(a,b)} \underset{\text{Def norm}}{=} \int_a^b |\Pi_h f(x) - f(x)|^p \, dx =$$

$a = x_0 < x_1 < \dots$

$$= \sum_{j=0}^{m} \int_{x_j}^{x_{j+1}} |\Pi_h f(x) - f(x)|^p \, dx =$$

$$\underset{\text{Def norm}}{=} \sum_{j=0}^{m} \|\Pi_h f - f\|^p_{L^p(x_j, x_{j+1})} \leq$$

On (x_j, x_{j+1}) $\Pi_h f$ is linear by Def

$$\leq \sum_{j=0}^{m} (C \cdot h^2)^p \, \|f''\|^p_{L^p(x_j, x_{j+1})}$$

error in linear interpol. see previous theorem

$$\leq (C h^2)^p \sum_{j=0}^{m} \|f''\|^p_{L^p(x_j, x_{j+1})}$$

$$\leq (C \cdot h^2)^p \cdot \|f''\|^p_{L^p(a,b)}$$

$\Rightarrow \quad \| \Pi_h f - f \|_{L^p(a,b)} \leq C \cdot h^2 \, \| f'' \|_{L^p(a,b)}$

;-) $\boxtimes$

<u>Ex:</u> Consider $f(x) = (x-1)(x-2) + 1$ on $[1,3]$

$= x^2 - 3x + 3$

and $\Pi_h f \in V_h(1,3)$.

Find bounds for the errors

$\| \Pi_h f - f \|_{L^p(a,b)}$ for $p = 1, 2, \infty$.

The theorem tells us that

$$\| \Pi_h f - f \|_{L^p(a,b)} \leq C \cdot h^2 \cdot \underbrace{\| f'' \|_{L^p(a,b)}}_{\text{compute this!}}$$

$p = 1$: $C \cdot h^2 \cdot 4$

$p = 2$: $C \cdot h^2 \cdot \sqrt{2} \cdot 2$, $p = \infty$: $C \cdot h^2 \cdot 2$

3) Lagrange interpolation

Recall: $a = x_0 < x_1 < x_2 < \ldots < x_q = b$

Lagrange polyn. $\lambda_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{q} \frac{x - x_j}{x_i - x_j}$

for $i = 0, 1, 2, \ldots, q$.

$\mathbb{P}^{(q)}(a,b) = \operatorname{span}(\lambda_0, \lambda_1, \ldots, \lambda_q)$ and

$$p(x) = \sum_{j=0}^{q} p(x_j) \cdot \lambda_j(x) \quad \text{for } x \in [a,b]$$

and any $p \in \mathbb{P}^{(q)}(a,b)$.

Def: Let $f : [a,b] \to \mathbb{R}$ continuous,

Lagrange interpolant is given by

$$\Pi_q f(x) = \sum_{j=0}^{q} f(x_j) \lambda_j(x)$$

Ex: Consider $f(x) = x^3 + 1$ for $x \in [0,2]$.

What are $\Pi_1 f$ and $\Pi_2 f$?

(i) For $\Pi_1 f$, the linear interpolant, we have

$$n = 1 \quad \text{and} \quad \begin{array}{ccc} x_0 & < & x_1 \\ \shortparallel & & \shortparallel \\ 0 & & 2 \end{array}$$

By def, we have

$$\Pi_1 f(x) = f(x_0) \cdot \lambda_0(x) + f(x_1) \cdot \lambda_1(x)$$

We have $f(x_0) = f(0) = 1$ and $f(x_1) = f(2) = 9$

By def of Lagrange polyn:

$$\lambda_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-2}{-2} = 1 - \frac{x}{2}$$

$$\lambda_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{2 - 0} = \frac{x}{2}$$

$$\hookrightarrow \Pi_1 f(x) = 1 \cdot \left(1 - \frac{x}{2}\right) + 9 \cdot \frac{x}{2} = \ldots = 4x + 1$$

(ii) $\Pi_2 f(x) = \ldots = 3x^2 - 2x + 1$

Hint: $\Pi_2 f(x) = f(x_0) \cdot \lambda_0(x) + f(x_1) \cdot \lambda_1(x) + f(x_2)\lambda_2(x)$, where

$$\begin{array}{ccccc} x_0 & < & x_1 & < & x_2 \\ \| & & \| & & \| \\ 0 & & 1 & & 2 \end{array}$$