Recall: • Galerkin FEM for BVP:

$$\begin{cases} -u''(x) = f(x) & 0<x<1 \quad \text{DE} \\ u(0)=0,\ u(1)=0 & \quad \text{BC} \end{cases}$$

(VF) Find $u \in V^0$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in V^0$!

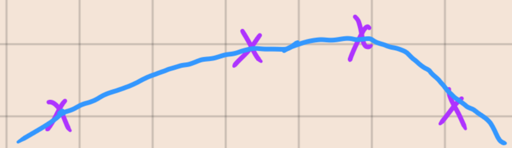
(FE) Find $U \in V_h^0$ s.t. $(U', \chi')_{L^2} = (f, \chi)_{L^2} \quad \forall \chi \in V_h^0$!

Here $U(x) := \sum_{j=1}^{m} \xi_j \varphi_j(x)$ and

Linear system of eq. $A\vec{\xi} = \vec{b}$

$$b_i = \underbrace{\int_0^1 f(x) \overset{\text{Hat fct.}}{\varphi_i(x)}\, dx}_{=?} \quad \text{for } i = 1, \dots, m$$

• Interpolation



x data sets $(x_j, f(x_j))$

$\Pi_h f$, $\Pi_q f$

4) Numerical integration / quadrature rules:

Prob: Find a numerical approximation to integrals

$$\int_a^b f(x)\,dx \approx\ ?$$

<u>Idea!</u> We can approximate $f(x)$ by a polynomial (or an interpolant polyn.)

$$f(x) \approx \underbrace{\Pi_q f(x)}_{\text{polyn. degree} \leq q}$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \underbrace{\int_a^b \Pi_q f(x)\,dx}_{\text{easier to compute!}}$$

We now present 3 classical quadrature formulas (QF):

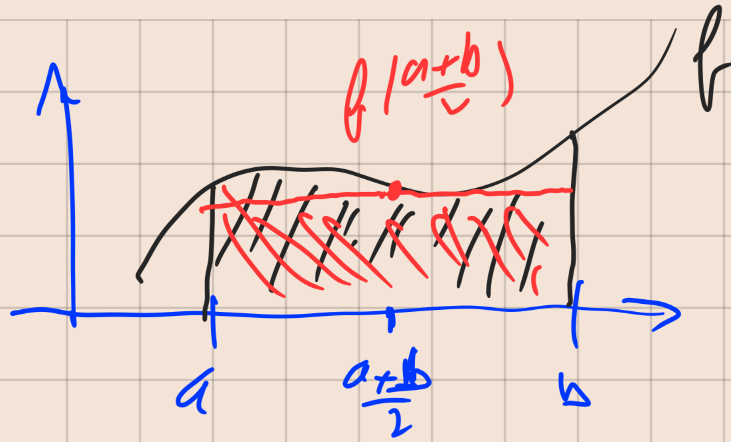
(i) $q=0 \rightsquigarrow$ approx. $f(x)$ by a constant polyn.:

$$f(x) \approx f\left(\frac{a+b}{2}\right)$$

We get $\int_a^b f(x)\,dx \approx \int_a^b \underbrace{f\left(\frac{a+b}{2}\right)}_{\text{constant}}dx \approx f\left(\frac{a+b}{2}\right)(b-a)$

$\Rightarrow$ The midpoint rule reads

$$\int_a^b f(x)\,dx \approx (b-a)\,f\left(\frac{a+b}{2}\right)$$



(ii) $q=1 \rightsquigarrow$ approx $f(x) \approx \underbrace{f(a)\,\lambda_0(x) + f(b)\cdot\lambda_1(x)}_{\Pi_1 f(x) \text{ Lagrange interpolant of degree 1}}$

where $\lambda_0(x) = \frac{x-b}{a-b}$, $\lambda_1(x) = \frac{x-a}{b-a}$

Hence, we get the approx:

$$\int_a^b f(x)dx \approx \int_a^b f(a)\,\lambda_0(x)\,dx + \int_a^b f(b)\,\lambda_1(x)\,dx \approx$$

$$\approx \frac{f(a)}{a-b}\int_a^b (x-b)\,dx + \frac{f(b)}{b-a}\int_a^b (x-a)\,dx \approx$$

$$\approx \frac{f(a)}{a-b}\left.\frac{(x-b)^2}{2}\right|_a^b + \frac{f(b)}{b-a}\left.\frac{(x-a)^2}{2}\right|_a^b \approx$$

$$\approx \frac{f(a)}{a-b}\left(-\frac{(a-b)^2}{2}\right) + \frac{f(b)}{b-a}\,\frac{(b-a)^2}{2} \approx$$

$$\approx f(a)\,\frac{b-a}{2} + f(b)\,\frac{b-a}{2} \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$$

↳ The trapezoidal rule reads

$$\int_a^b f(x)\,dx \approx \underbrace{\frac{b-a}{2}\left(f(a)+f(b)\right)}$$

(Babylon knows it)

area of trapezoidal

(iii) $q = 2 \rightarrow \ldots \rightsquigarrow$

Simpson's rule :

↳ not exact!

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$5+1$

(details Book)

Ex: Use the above QF to approximate the integral

$$\int_0^{\pi/4} \sin(x)\,dx \approx ?$$

(i) Midpoint gives

$$\int_0^{\pi/4} \sin(x)\,dx \approx \left(\frac{\pi}{4}-0\right)\cdot \sin\left(\frac{\pi/4+0}{2}\right)$$

$$\approx \frac{\pi}{4}\cdot \sin\left(\frac{\pi}{8}\right) \approx 0,3$$

(ii) Trapezoidal rule $\approx 0,27$

(iii) Simpson's rule $\approx 0{,}2929$

Exact sol.: $\int_0^{\pi/4} \sin(x)\,dx \approx 0{,}29289$

⚠ <u>Rem</u>: In practice, one considers first a partition of $[a,b]$ and <u>then</u> apply a QF:

expl. midpoint

$$\int_a^b f(x)\,dx \underset{\uparrow}{=} \sum_{j=0}^{N-1} \underbrace{\int_{x_j}^{x_{j+1}} f(x)\,dx}_{\text{apply QF}} \approx \sum_{j=0}^{N-1} h f\left(\frac{x_{j+1}+x_j}{2}\right)$$

$a = x_0 < x_1 < x_2 < \dots < x_N = b$

$x_{j+1} - x_j = h$ (small)

$$\approx \sum_{j=0}^{N-1} h \cdot f\left(\frac{x_{j+1}+x_j}{2}\right)$$

Chapter V: FEM for two-point BVP

Goal: Use all the above theoretical / practical tools to study and analyse FEM for various BVP

1) Finite element method:

Let us derive a FEM for the BVP

$$(BVP)\begin{cases} -(a(x)u'(x))' = f(x) & \text{for } 0<x<1 \\ u(0)=0, \ u(1)=0 & \end{cases}$$

Here $a(x)$ is given (>0) and $f(x)$ given.

The above is a simple model for a load on a bar:

$u(x) \rightsquigarrow$ displacement of bar

bar of length 1

$a(x)$ is the modulus of elasticity

(i) In order to find a (VF) of the (BVP),

we consider the space

$V^0 = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1), \; v(0) = v(1) = 0 \}$

hom. Dirichlet

Then, multiply (BVP) with test function $v \in V^0$, integrate and find:

$$-\int_0^1 (a(x)u'(x))' v(x)\,dx = \int_0^1 f(x) v(x)\,dx$$

Integration by part gives:

$$\underbrace{-\big(a(x)u'(x)\big)v(x)\Big|_0^1}_{\underbrace{-a(1)u'(1)v(1) + a(0)u'(0)v(0)}_{=0 \text{ since } v \in V^0}} + \int_0^1 a(x)u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

The variational formulation of (BVP) is:

(VF) Find $u \in V^0$ s.t. $\int_0^1 a(x)\, u'(x)\, v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx$

trial space

$\forall v \in V^0$

test space

Obs: Here, test = trial

(ii) To get a (FE) problem, we consider the space

piecewise

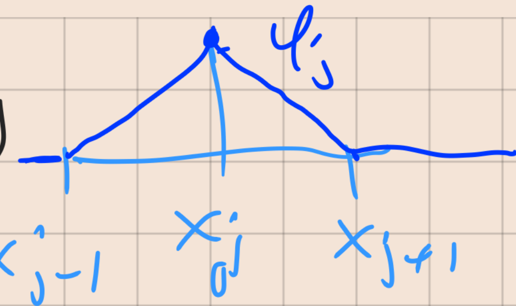
$V_h^0 = \{ v : [0,1] \to \mathbb{R} : v \text{ is cont., pw linear on } T_h \}$,

where

T_h uniform partition of $[0,1]$:

$$0 = x_0 < x_1 < x_2 < \dots < x_{m+1} = 1$$

and $h = x_{j+1} - x_j$

Using hat fct, $\varphi_j(x)$ ,

We get $V_h^0 = \mathrm{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$

no need of φ_0, φ_{m+1}
since Dirichlet BC

$\dim(V_h^0) = m$,

The FE problem reads

(FE) Find $U \in V_h^0$ s.t.

$$\int_0^1 a(x)\, U'(x)\, \chi'(x)\, dx = \int_0^1 f(x)\, \chi(x)\, dx$$

$$\forall \chi \in V_h^0$$

(iii) In order to get a linear system of equations from (FE), we choose test fct $\chi(x) = \varphi_i(x)$ for $i = 1, 2, \dots, m$ and write

$$\underset{\text{A}}{U(x)} = \sum_{j=1}^{m} \zeta_j \varphi_j(x) .$$

φ_j → basis

ζ_j → coordinates (unknow)

$V_h^0 = \mathrm{span}(\varphi_1, \dots, \varphi_m)$

Insert the above into (FE) to get:

$$\int_0^1 a(x) \sum_{j=1}^{m} \zeta_j \varphi_j'(x) \varphi_i'(x) \, dx = \int_0^1 f(x) \varphi_i(x) \, dx$$

$$\forall i = 1, \dots, m$$

$\Leftrightarrow$

$$\sum_{j=1}^{m} \zeta_j \underbrace{\int_0^1 a(x) \varphi_j'(x) \varphi_i'(x) \, dx}_{S_{ij}} = \underbrace{\int_0^1 f(x) \varphi_i(x) \, dx}_{b_i}$$

$\Leftrightarrow \quad S' \cdot \vec{\zeta} = \vec{b}$, where

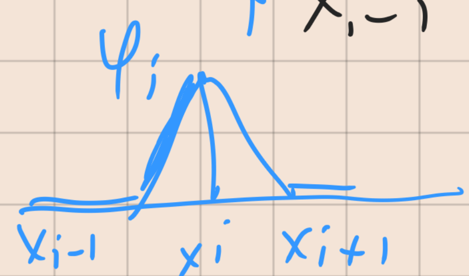
the matrix S' $(m \times m)$ is called
stiffness matrix

the vector $\vec{b}$ $(m \times 1)$ is the
load vector

the vector $\vec{\zeta}$ $(m \times 1)$ is the vector
of unknowns.

As done before, we can compute
the entries of matrix S':

$$s_{ii} = \int_0^1 a(x) \left(\varphi_i'(x)\right)^2 dx = \int_{x_{i-1}}^{x_{i+1}} a(x) \left(\varphi_i'(x)\right)^2 dx =$$



$$= \frac{1}{h^2} \int_{x_{i-1}}^{x_i} a(x)\,dx + \frac{1}{h^2} \int_{x_i}^{x_{i+1}} a(x)\,dx$$

def $\varphi_i(x)$

use QF if cannot find an exact solution

Similar computations can be done for S_{ij} and b_i ...

(book)

2) FEM for convection-diffusion-absorbtion BVP

The main ideas of the previous section needs some adaptation when changing BC / DE:

For instance, consider

$$\text{(BVP)} \quad \begin{cases} -u''(x) + 4u(x) = 0 & \text{for } 0 < x < 1 \\ \underline{u(0) = \alpha} \text{ and } \underline{u(1) = \beta} \end{cases},$$

where $\alpha, \beta \in \mathbb{R}^{*}$ given

Obs: These BC are called nonhomogeneous Dirichlet BC

$(\alpha, \beta \neq 0)$