Recall:

(BVP) $\begin{cases} -u''(x) + u'(x) = 1 & \text{for } 0<x<1 \\ u(0)=0,\ u'(1)=\beta \neq 0 & \text{non-hom. Neumann BC} \end{cases}$

$\Updownarrow$

(VF)

$\downarrow$

(FE) Find $U \in V_h$ s.t. $(U', \chi')_{L^2} + (U', \chi)_{L^2} = \int_0^1 \chi(x)\,dx + \beta \chi(1)$ $\quad \forall \chi \in V_h$

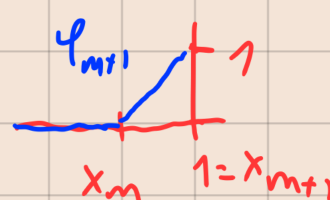
$\downarrow$

lin. syst. of eq.?

where $V_h = \text{span}(\varphi_1, \dots, \varphi_{m+1})$, $T_h: x_0 = 0 < x_1 < x_2 < \dots < x_m < x_{m+1} = 1$

Choose $\chi = \varphi_i$ for $i = 1, 2, \dots, m+1$ and write

$$U(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$$



In order to find a matrix / vector for a linear system of eq., we insert the choice of test fct. $\chi = \varphi_i$ and $U(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$ into the FE problem and get

$$\Big(\sum_{j=1}^{m+1} \xi_j \varphi_j', \varphi_i'\Big)_{L^2} + \Big(\sum_{j=1}^{m+1} \xi_j \varphi_j', \varphi_i\Big)_{L^2} =$$

$$= \int_0^1 \varphi_i(x)\,dx + \beta \varphi_i(1)$$

Use linearity of inner product:

$$\sum_{j=1}^{m+1} \zeta_j \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{S_{ij}} + \sum_{j=1}^{m+1} \zeta_j \underbrace{(\varphi_j, \varphi_i)_{L^2}}_{C_{ij}} = b_i,$$

where $b_i = \int_0^1 \varphi_i(x)\,dx + \beta\varphi_i(1)$ for $i = 1, 2, \dots, m+1$.

This gives us the following system:

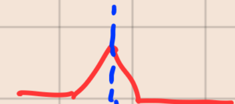
$$S \cdot \vec{\zeta} + C \cdot \vec{\zeta} = \vec{b},$$

where

S is $(m+1) \times (m+1)$ stiffness matrix

$$S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & & 2 \\ 0 & & -1 & \boxed{1} \end{pmatrix}$$

φ_{m+1} $\frac{1}{2}$



and

G' is $(m+1)\times(m+1)$ convection matrix

$$G' = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & & 0 \\ -1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ 0 & & & & -1 & 1 \end{pmatrix}$$

and

$\vec{b}$ is $(m+1)\times 1$ vector

$$\vec{b} = \dots = \begin{pmatrix} h \\ h \\ \vdots \\ h \\ \underbrace{\boxed{\frac{h}{2}}}_{\varphi_{m+1}\text{ half hat}} + \underbrace{\boxed{\beta}}_{\text{Neumann BC.}} \end{pmatrix}$$

<u>Rem</u>: Observe that S' and also mass matrix M are symmetric but G' is not.

3) BVP $\Longleftrightarrow$ VF:

Here, we consider the problem (f continuous and bounded)

$$(BVP) \begin{cases} -u''(x) = f(x) & 0 < x < 1 \\ u(0) = 0 = u(1) & \text{(hom. Dirichlet BC)} \end{cases}$$

and show that it is equivalent to the variational formulation

(VF) Find $u \in V^0$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in V^0$,

where $V^0 = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \text{ and } v(0) = v(1) = 0 \}$

Th: "(BVP)" $\Leftrightarrow$ "(VF) and $u \in C^2(0,1)$"

equivalent

2 times cont. diff.

Proof:

$\Rightarrow$: Already done!

Multiply (BVP) with test fct $v \in V^0$ and integrate to get

$$-\underbrace{\int_0^1 u''(x)v(x)\,dx}_{\underbrace{-u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v\in V^0} + \int_0^1 u'(x)v'(x)\,dx \text{ (int. by part)}} = \int_0^1 f(x)v(x)\,dx \quad \forall v \in V^0$$

$$\Rightarrow \quad (u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in V^0$$

which is (VF)!

$\Leftarrow$: We start with (VF): $(u',v')_{L^2} = (f,v)_{L^2}$

Integrating by part gives

$$\underbrace{-u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v\in V^0} - \int_0^1 u''(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx$$

$$\Rightarrow \int_0^1 (u''(x)+f(x))\,v(x)\,dx = 0 \quad \forall v\in V^0 \quad (*)$$

We show by contradiction that this implies (BVP) $u''(x)+f(x)=0$ for $0<x<1$.

We suppose that $u''+f\neq 0$.

By continuity (of u'' and f) $\exists\, x\in(0,1)$ s.t. $u''+f\neq 0$ in a neighborhood of x.

If we choose v s.t. it has the same sign as $u''+f$ in this neighborhood and is zero elsewhere, then $\int_0^1 (u''(x)+f(x))\,v(x)\,dx > 0$

This is a contradiction to (*).

Therefore $u''(x) + f(x) = 0$ on $(0,1)$!!

I.e. (BVP) ok ;-)

$\blacksquare$

<u>Rem.</u> The space V^0 above is sometimes also denoted by

$$H_0^1(0,1).$$

$$H_0^1(a,b) = \{ v : [a,b] \to \mathbb{R} : v, v' \in L^2(a,b), \; v(a) = v(b) = 0 \}$$

4) Error estimates:

Th: (Poincaré inequality)

Let $L > 0$ and $\Omega = (0, L)$. Assume that $u \in H_0^1(\Omega)$. Then,

control $\qquad\qquad\qquad\qquad \leftarrow$ control

$$\|u\|_{L^2(\Omega)} \le \underbrace{C_L}_{} \cdot \|u'\|_{L^2(\Omega)}$$

constant only depends on L

Proof:

For $u \in H_0^1(\Omega)$, one has

$$⚠ \quad \underbrace{u(x) = \int_0^x u'(\xi)\, d\xi}_{u(x) - u(0) = u(x) - 0 = u(x).} \qquad \forall x \in \Omega.$$

We look at

$$|u(x)|^2 = \left| \int_0^x 1 \cdot u'(\xi) d\xi \right|^2 \overset{C-S}{\le}$$

$$\leq \left(\int_0^x 1^2 d\xi\right) \cdot \left(\int_0^x (u'(\xi))^2 d\xi\right)$$

$$\leq x \cdot \int_0^{x'} (u'(\xi))^2 d\xi \leq x \cdot \int_0^L (u'(\xi))^2 d\xi$$

$x \leq L$

We integrate:

$$\int_0^L |u(x)|^2 dx \leq \int_0^L x\,dx \cdot \int_0^L |u'(\xi)|^2 d\xi \Leftrightarrow$$

$$\|u\|^2_{L^2(\Omega)} \leq \frac{L^2}{2} \cdot \|u'\|^2_{L^2(\Omega)} \quad :-)$$

(Def norm) (Def norm)

$\blacksquare$

<u>Def</u>: For $f, g \in H_0^1(a,b)$, we define the energy inner product

$$(f,g)_E = \int_a^b f'(x) g'(x) dx = (f', g')_{L^2(a,b)}$$

and the energy norm

$$\|f\|_E = \sqrt{(f,f)_E} = \sqrt{(f',f')_{L^2}}$$

We shall now estimate the error in the energy norm of FEM.

Recall:

$$(BVP)\begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0)=u(1)=0 \end{cases}$$

(VF) Find $u \in V^0 = H_0^1$ s.t. $(u',v')_{L^2} = (f,v)_{L^2}$ $\forall v \in V^0$

(FE) Find $U \in V_h^0$ s.t. $(U',\chi')_{L^2} = (f,\chi)_{L^2}$ $\forall \chi \in V_h^0$

$V_h^0 = \mathrm{span}(\varphi_1, \varphi_2, \dots, \varphi_m)$.

Observe that $V_h^0 \subset V^0$, thus one can take $v = \chi$ in (VF) ⚠

We thus get

$$(u', \chi')_{L^2} = (f, \chi)_{L^2} \qquad (VF)$$

$$(U', \chi')_{L^2} = (f, \chi)_{L^2} \qquad (FE)$$

Difference:

$$(u', \chi')_{L^2} - (U', \chi')_{L^2} = 0 \qquad \underline{\text{or}}$$

$$(u' - U', \chi')_{L^2} = 0 \qquad \forall \chi \in V_h^0 \quad \underline{\text{or}}$$

$$(u - U, \chi)_E = 0 \qquad \forall \chi \in V_h^0$$

Galerkin orthogonality

Error of FEM is orthogonal to V_h^0 in the energy inner product.

With the above, we show

Th: Let u be the sol. (BVP)/(VF) and U be the FE sol. Then,

$$\|u-U\|_E \leq \|u-\chi\|_E \quad \forall \chi \in V_h^0$$

FE approx. of u in V_e^0 is the best approx. of u in the energy norm!!

Proof:

We compute

$$\|u-U\|_E^2 \underset{\text{Def norm}}{=} (u-U, u-U)_E = (u-U, u-\chi+\chi-U)_E$$

$$= (u-U, u-\chi)_E + \underbrace{(u-U, \underbrace{\chi-U}_{\in V_h^0})_E}_{=0 \text{ Galerkin } \perp} =$$

$$= (u-U, u-\chi)_E \overset{C-S}{\leq} \|u-U\|_E \cdot \|u-\chi\|_E$$

$$\Rightarrow \|u-U\|_E \leq \|u-\chi\|_E \qquad ;-) \qquad \blacksquare$$

We conclude with

Th.: (a priori error estimate)

Let u be sol. to (VF) and U be FE sol. Assume $u \in C^2(0,1)$, then $\exists\, C>0$ s.t.

$\mathcal{C}^2(0,1)$

$$\|u - U\|_E \leq C \cdot h \cdot \|u''\|_{L^2(0,1)}$$

<u>Proof</u>:

previous theorem

$$\|u - U\|_E \leq \|u - \underbrace{\Pi_h u}_{b}\|_E = \|u' - (\Pi_h u)'\|_{L^2} \leq$$

cont. pw linear interpolant of u

$$\leq C \cdot \boxed{h} \cdot \|u''\|_{L^2}$$:-)

Chapter IV

$\left(h = \frac{1}{m+1}\right)$

$\blacksquare$