Matematik Chalmers

TMA683 Tillämpad matematik K2/Bt2

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Hjälpmedel: Endast tabell på backsidan av testen. Kalkylator ej tillåten.
Betygsgränser, **3**: 20–29p, **4**: 30–39p och **5**: 40–50p.

- För full poäng på en uppgift krävs en korrekt och välmotiverad lösning. Enbart svar/resultatet av en beräkning utan motivering ger inga poäng på uppgiften.
- Svaren skall, om möjligt, anges exakt och förenklade på lämpligt sätt.

Lösningar/Granskning: Se kurshemsidan.

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1. Använd Laplacetransformer för att lösa differentialekvationen (8p)

$$y''(t) + 2y'(t) + 3y(t) = 3t, \qquad y(0) = 0, \quad y'(0) = 1.$$

2. (a) Bestäm L_2-projektionen av $f(x) = \pi \sin(\pi x)$ i $\mathcal{P}^{(1)}(0, 1)$. (4p)

(b) Bestäm $\|f\|_{L_2(0,1)}$ och $\|f\|_{L_\infty(0,1)}$. (4p)

3. (a) För vilka värden på $a > 0$ är funktionerna $\sin(ax)$ och $\cos(ax)$ ortogonala i $L_2(0, 1)$? (4p)

(b) Bestäm Fourier cosine-serien med perioden 2π till funktionen $f(x) = \sin x,\ 0 \le x \le \pi$. (5p)

4. Betrakta den inhomogena vågekvationen (10p)

$$\begin{cases} \ddot{u}(x,t) - u''(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0,\ u(1,t) = 1, & \qquad\qquad\quad t > 0, \\ u(x,0) = 2x & 0 < x < 1 \\ \dot{u}(x,0) = 0 & 0 < x < 1 \end{cases}$$

Använd variabelseparationsmetoden för att bestämma $u(x, t)$.

5. Härled *variationsformulering* för begynnelsevärdesproblemet (a, b, α, β är icke-nollkonstanter),

$$\begin{cases} -u'' + au' + bu = f, & 0 < x < 1, \\ u'(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

(5p)

6. Betrakta begynnelsevärdesproblemet

$$\begin{cases} -u'' + 2u = 3, & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

(a) Härled *variationsformulering.* (3p)

(b) Härled cG(1) finita element formulering (kontinuerliga styckvis linjära polynomer). Härled det linjära ekvationssystemet på formen $S\xi + M\xi = F$. Beräkna styvhetsmatrisen S (Stiffness matrix) och lastvektorn F (Load vector). Beräkna **ej** massmatrisen M (Mass matrix). (7p)

(OBS! Använd likformig partition med steglängd h och $\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$.)

LYCKA TILL! \FS

Table of Laplace Transforms and trigonomerty

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin bt$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin bt - bt\cos bt)$	$\frac{1}{(s^2+b^2)^2}$
$2\sin a\sin b \ = \cos(a-b)-\cos(a+b)$	
$2\sin a\cos b \ = \sin(a-b)+\sin(a+b)$	
$2\cos a\cos b \ = \cos(a-b)+\cos(a+b)$	

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1. Take the Laplace transform

$$s^2Y(s) - sy(0) - y'(0) + 2\big(sY(s) - y(0)\big) + 3Y(s) = \frac{3}{s^2}$$

$$\Longrightarrow \quad (s^2+2s+3)Y(s) = 1 + \frac{3}{s^2} \qquad \Longrightarrow \quad Y(s) = \frac{s^2+3}{s^2(s^2+2s+3)}$$

Now we split the term on the right side:

$$\begin{aligned} Y(s) &= \frac{s^2+3}{s^2(s^2+2s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+3} \\ &= \frac{(A+C)s^3 + (2A+B+D)s^2 + (3A+2B)s + 3B}{s^2(s^2+2s+3)} \end{aligned}$$

$$\Longrightarrow \begin{cases} A+C=0 \\ 2A+B+D=1 \\ 3A+2B=0 \\ 3B=3 \end{cases} \quad \Longrightarrow \quad A = -\frac{2}{3}, \quad B=1, \quad C=\frac{2}{3}, \quad D=\frac{4}{3}$$

Therefore

$$\begin{aligned} Y(s) &= -\frac{2}{3}\frac{1}{s} + \frac{1}{s^2} + \frac{2}{3}\frac{s+2}{s^2+2s+3} = -\frac{2}{3}\frac{1}{s} + \frac{1}{s^2} + \frac{2}{3}\frac{(s+1)+1}{(s+1)^2+2} \\ &= -\frac{2}{3}\frac{1}{s} + \frac{1}{s^2} + \frac{2}{3}\frac{(s+1)}{(s+1)^2+2} + \frac{\sqrt{2}}{3}\frac{\sqrt{2}}{(s+1)^2+2} \\ \Longrightarrow \quad y(t) &= \mathcal{L}^{-1}\{Y(s)\} = -\frac{2}{3} + t + \frac{2}{3}e^{-t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-t}\sin\sqrt{2}t \end{aligned}$$

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2. (a) $Pf \in \mathcal{P}^{(1)}(0,1)$, so $Pf(x) = \xi_0 + \xi_1 x,\ x \in (0,1)$. To find the unknowns $\xi_0,\ \xi_1$ we need two equations. By the definition of L_2-projection:

$$\int_0^1 f(x)x^i\ dx = \int_0^1 (Pf)(x)x^i\ dx, \quad i = 0,1$$

so

$$\begin{cases} \int_0^1 \pi\sin(\pi x)\ dx = \int_0^1 (\xi_0 + \xi_1 x)\ dx \\ \\ \int_0^1 \pi\sin(\pi x)x\ dx = \int_0^1 (\xi_0 + \xi_1 x)x\ dx \end{cases} \Longrightarrow \begin{cases} \xi_0 + \frac{1}{2}\xi_1 = 2 \\ \\ \frac{1}{2}\xi_0 + \frac{1}{3}\xi_1 = 1 \end{cases} \Longrightarrow \xi_0 = 2, \quad \xi_1 = 0$$

Hence $\quad (Pf)(x) = 2.$

(b)

$$\|f\|_{L_2(0,1)} = \sqrt{\int_0^1 |f(x)|^2\ dx} = \sqrt{\int_0^1 \pi^2\sin^2(\pi x)\ dx} = \sqrt{\pi^2\int_0^1 \frac{\big(1-\cos(2\pi x)\big)}{2}\ dx} = \frac{\pi\sqrt{2}}{2}$$

$$\|f\|_{L_\infty(0,1)} = \sup_{x\in(0,1)} |f(x)| = \sup_{x\in(0,1)} |\pi\sin(\pi x)| = \pi$$

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3. (a) We should have

$$0 = \langle \sin(ax), \cos(ax)\rangle_{L_2(0,1)} = \int_0^1 \sin(ax)\cos(ax)\ dx = \frac{1}{2}\int_0^1 \sin(2ax)\ dx = -\frac{1}{4a}\big(\cos(2a) - 1\big)$$

$$\implies \cos(2a) = 1 \implies 2a = 2n\pi \implies a = n\pi,\quad n = 1, 2, \dots$$

(b) For the Fourier cosine series we have $b_n = 0,\ n = 1, \dots$ and

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi \sin x\ dx = \frac{4}{\pi},\\
a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx\ dx = \frac{1}{\pi}\int_0^\pi \sin(1-n)x\ dx + \frac{1}{\pi}\int_0^\pi \sin(1+n)x\ dx\\
&\overset{n\neq 1}{=} \frac{1}{\pi}\Big(\frac{-1}{1-n}\cos(1-n)x\Big]_0^\pi + \frac{1}{\pi}\Big(\frac{-1}{n+1}\cos(n+1)x\Big]_0^\pi\\
&= \frac{1}{\pi(n-1)}\Big(\cos(n-1)\pi - \cos 0\Big) + \frac{-1}{\pi(n+1)}\Big(\cos(n+1)\pi - \cos 0\Big)\\
&= \frac{(-1)^{n-1} - 1}{\pi(n-1)} + \frac{(-1)^{n+2} + 1}{\pi(n+1)} = -\frac{(-1)^n + 1}{\pi(n-1)} + \frac{(-1)^n + 1}{\pi(n+1)}\\
&= -\frac{2\big((-1)^n + 1\big)}{\pi(n^2 - 1)} = \frac{2\big((-1)^{n-1} - 1\big)}{\pi(n^2 - 1)}
\end{aligned}$$

And for $n = 1$ we have

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\ dx = \frac{1}{\pi}\int_0^\pi \sin 2x\ dx = 0$$

Hence, for $0 \le x \le \pi$,

$$\begin{aligned}
f(x) &= \frac{4}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{\big((-1)^{n-1} - 1\big)}{n^2 - 1}\cos nx\\
&= \frac{4}{\pi} - \frac{4}{\pi}\Big(\frac{1}{2^2 - 1}\cos 2x + \frac{1}{4^2 - 1}\cos 4x + \cdots\Big) = \frac{4}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{(2k)^2 - 1}
\end{aligned}$$

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4. We look for the solution as $u(x,t) = v(x,t) + s(x)$. Putting the solution in the PDE, we have

$$\begin{cases}
\ddot{v}(x,t) - v''(x,t) - s''(x) = 0, & 0 < x < 1,\quad t > 0,\\
v(0,t) + s(0) = 0,\ v(1,t) + s(1) = 1, & \qquad\qquad\ \ t > 0,\\
v(x,0) + s(x) = 2x & 0 < x < 1\\
\dot{v}(x,0) = 0 & 0 < x < 1
\end{cases}$$

So we need to solve an ODE and a PDE:

$$\begin{cases}
-s''(x) = 0, & 0 < x < 1,\\
s(0) = 0,\ s(1) = 1.
\end{cases}$$

$$\begin{cases}
\dot{v}(x,t) - v''(x,t) = 0, & 0 < x < 1,\quad t > 0,\\
v(0,t) = 0,\ v(1,t) = 0, & \qquad\qquad\ \ t > 0,\\
v(x,0) = 2x - s(x) & 0 < x < 1\\
\dot{v}(x,0) = 0 & 0 < x < 1
\end{cases}$$

First we solve the ODE:

$$\begin{cases}
s(x) = C_1 x + C_2\\
s(0) = 0,\ s(1) = 1
\end{cases}
\overset{s(0)=0}{\underset{s(1)=1}{\Longrightarrow}}
\begin{cases}
C_2 = 0\\
C_1 + C_2 = 1\ \Rightarrow C_1 = 1
\end{cases}
\quad \Rightarrow\ s(x) = x$$

Now, we solve the homogeneous PDE, that is

$$\begin{cases}
\dot{v}(x,t) - v''(x,t) = 0, & 0 < x < 1,\quad t > 0,\\
v(0,t) = 0,\ v(1,t) = 0, & \qquad\qquad\ \ t > 0,\\
v(x,0) = 2x - s(x) = x & 0 < x < 1\\
\dot{v}(x,0) = 0 & 0 < x < 1
\end{cases}$$

We look for the solution $v(x,t) = X(x)T(t)$. Then

$$X\ddot{T} - X''T = 0 \implies \frac{\ddot{T}}{T}(t) = \frac{X''}{X}(x) = \lambda \overset{\lambda=-\mu^2}{\implies} \begin{cases} X''(x) = -\mu^2 X(x) \\ \ddot{T}(t) = -\mu^2 T(t) \end{cases}$$

For the first equation, considering the homogeneous boundary conditions, we have

$$\begin{cases} X''(x) = -\mu^2 X(x) \\ X(0) = X(1) = 0 \end{cases} \implies \begin{cases} X(x) = A\cos\mu x + B\sin\mu x \\ X(0) = X(1) = 0 \end{cases}$$

$X(0) = 0 \;\Rightarrow\; A = 0 \;\Rightarrow\; X(x) = B\sin\mu x$

$X(1) = 0 \;\Rightarrow\; B\sin\mu = 0 \overset{B\neq 0}{\implies} \sin\mu = 0 \;\Rightarrow\; \mu = n\pi,\; n = 1, 2, \cdots$

So $X_n(x) = B_n \sin(n\pi x)$.

For the second equation $\ddot{T}(t) = -\mu^2 T(t) = -(n\pi)^2 T(t)$, we have:

$T_n(t) = A_n \cos(n\pi t) + B_n \sin(n\pi t)$.

Hence, by superposition principle, solution is

$$v(x,t) = \sum_{n=1}^{\infty} \Big(A_n \cos(n\pi t) + B_n \sin(n\pi t)\Big) \sin(n\pi x)$$

and, we note that,

$$\dot{v}(x,t) = \sum_{n=1}^{\infty} \Big(-n\pi A_n \sin(n\pi t) + n\pi B_n \cos(n\pi t)\Big) \sin(n\pi x)$$

Now, using the initial conditions $v(x,0) = x$ and $\dot{v}(x,0) = 0$, we have

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = x$$

$$\dot{v}(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) = 0$$

and therefore

$$A_n = \frac{2}{1}\int_0^1 x\sin(n\pi x)\;dx = 2\big[-\frac{1}{n\pi}x\cos(n\pi x)\big]_0^1 + 2\frac{1}{n\pi}\int_0^1 \cos(n\pi x)\;dx = \frac{2(-1)^{n+1}}{n\pi}$$

$$n\pi B_n = 0 \;\Rightarrow\; B_n = 0$$

Hence

$$u(x,t) = v(x,t) + s(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\cos(n\pi t)\sin(n\pi x) + x$$

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5. Define function spaces

$$V = \{v \mid v, v' \in L_2(0,1),\; v(1) = \beta\}, \quad \tilde{V} = \{v \mid v, v' \in L_2(0,1),\; v(1) = 0\}$$

Now, multiply the differential equation by a test function $v \in \tilde{V}$, then integrate over $(0,1)$ and integrate by parts:

$$-\int_0^1 u''v\;dx + a\int_0^1 u'v\;dx + b\int_0^1 uv\;dx = \int_0^1 fv\;dx$$

$$\implies -u'(1)\underbrace{v(1)}_{=0} + \underbrace{u'(0)}_{=\alpha}v(0) + \int_0^1 u'v'\;dx + a\int_0^1 u'v\;dx + b\int_0^1 uv\;dx = \int_0^1 fv\;dx$$

Hence the variational formulation (VF) is:

Find $u \in V$, such that

$$\int_0^1 u'v'\;dx + a\int_0^1 u'v\;dx + b\int_0^1 uv\;dx = \int_0^1 fv\;dx - \alpha v(0), \qquad \forall v \in \tilde{V}$$

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6. (a) Define

$$V = \{v \mid v, v' \in L_2(0,1)\}.$$

Multiply the DE by a test function $v \in V$, then integrate over $(0,1)$ and integrate by parts:

$$-\int_0^1 u''v\ dx + 2\int_0^1 uv\ dx = \int_0^1 3v\ dx$$

$$\Longrightarrow\ -\underbrace{u'(1)}_{=0}v(1) + \underbrace{u'(0)}_{=0}v(0) + \int_0^1 u'v'\ dx + 2\int_0^1 uv\ dx = 3\int_0^1 v\ dx$$

VF (variational form) is:

$$\text{Find } u \in V, \text{ such that}$$

$$\int_0^1 u'v'\ dx + 2\int_0^1 uv\ dx = 3\int_0^1 v\ dx, \qquad \forall v \in V$$

(b) Consider a uniform partition with constant mesh size h:

$$\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$

The finite element space is

$$V_h = \{v \mid v \text{ is continuous p.w. linear on } \mathcal{T}_h\} = span\{\varphi_0, \varphi_1, \ldots, \varphi_m, \varphi_{m+1}\}$$

FEM (finite element method) is:

$$\text{Find } U \in V_h, \text{ such that}$$

$$\int_0^1 U'\chi'\ dx + 2\int_0^1 U\chi\ dx = 3\int_0^1 \chi\ dx, \qquad \forall \chi \in V_h$$

To write the matrix form:
choose $\quad \chi = \varphi_i,\ \ n = 0, 1, \ldots, m, m+1$
substitute $\quad U(x) = \sum_{j=0}^{m+1} \xi_j \varphi_j(x)$

$$\int_0^1 \Big(\sum_{j=0}^{m+1} \xi_j \varphi_j'(x)\Big)\varphi_i'(x)\ dx + 2\int_0^1 \Big(\sum_{j=0}^{m+1} \xi_j \varphi_j(x)\Big)\varphi_i(x)\ dx$$

$$= 3\int_0^1 \varphi_i(x)\ dx, \qquad i = 0, 1, \ldots, m, m+1$$

$$\Longrightarrow \sum_{j=0}^{m+1} \underbrace{\Big(\int_0^1 \varphi_j'(x)\varphi_i'(x)\ dx\Big)}_{=S_{i,j}}\xi_j + 2\sum_{j=0}^{m+1} \underbrace{\Big(\int_0^1 \varphi_j(x)\varphi_i(x)\ dx\Big)}_{=M_{i,j}}\xi_j$$

$$= \underbrace{3\int_0^1 \varphi_i(x)\ dx,}_{=F_i} \qquad i = 0, 1, \ldots, m, m+1$$

that is the linear system of equations (with $m+2$ unknowns $\xi_0, \xi_1, \ldots, \xi_m, \xi_{m+1}$),

$$S\xi + 2M\xi = F$$

For the stiffness matrix we have (note that φ_0 and φ_{m+1} are half hat functions):

<u>Stiffness matrix $S_{(m+2)\times(m+2)}$</u>:

$$S_{0,0} = \int_0^1 \varphi_0'\varphi_0'\ dx = \int_{x_0}^{x_1} \big(-\frac{1}{h}\big)^2\ dx = \frac{1}{h}$$

$$S_{m+1,m+1} = \int_0^1 \varphi_{m+1}'\varphi_{m+1}'\ dx = \int_{x_m}^{x_{m+1}} \big(\frac{1}{h}\big)^2\ dx = \frac{1}{h}$$

For $i = 1, \ldots, m$,

$$S_{i,i} = \int_0^1 \varphi_i' \varphi_i' \, dx = \int_{x_{i-1}}^{x_i} \big(\frac{1}{h}\big)^2 \, dx + \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)^2 \, dx = \frac{2}{h}$$

For $i = 0, \ldots, m-1$,

$$S_{i,i+1} = S_{i+1,i} = \int_0^1 \varphi_i' \varphi_{i+1}' \, dx = \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)\big(\frac{1}{h}\big) \, dx = -\frac{1}{h}$$

That is

$$S = \frac{1}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{(m+2)\times(m+2)}$$

<u>Load vector $F_{(m+2)\times 1}$:</u>

$$F_0 = 3 \int_0^1 \varphi_0 \, dx = 3 \int_{x_0}^{x_1} \varphi_0 \, dx = 3\frac{h}{2}$$

$$F_{m+1} = 3 \int_0^1 \varphi_{m+1} \, dx = 3 \int_{x_m}^{x_{m+1}} \varphi_{m+1} \, dx = 3\frac{h}{2}$$

For $i = 1, \ldots, m$,

$$F_i = 3 \int_0^1 \varphi_i \, dx = 3 \int_{x_{i-1}}^{x_{i+1}} \varphi_i \, dx = 3h$$

Hence

$$F = \frac{3}{2} h \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix}_{(m+2)\times 1}$$