Matematik Chalmers

TMA683 Tillämpad matematik K2/Bt2

2020–01–18; KL 8:30-12:30

Telefon: Kristian Holm: 031-772 5325; Examinator: Fardin Saedpanah 031-772 3515
Hjälpmedel: Endast tabell på backsidan av testen. Kalkylator ej tillåten.
Betygsgränser, **3**: 20–29p, **4**: 30–39p och **5**: 40–50p.
Lösningar/Granskning: Se kurshemsidan.

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1. Använd Laplacetransformer för att lösa differentialekvationen (7p)

$$\begin{cases} y'(t) + 2y(t) = e^{-3t}, & t > 0, \\ y(0) = 0. \end{cases}$$

2. (a) Visa att följande funktioner är linjärt oberoende (för $t \in \mathbb{R}$): (3p)

$$p_1(t) = (t-1)^2, \quad p_2(t) = (t-2)^2, \quad p_3(t) = (t-3)^2$$

(b) Bestäm den kontinuerlig styckvis linjära interpolanten $\pi_h f(x)$ (continuous piecewise linear interpolant) av funktionen $f(x) = 2x^2 - x$ då intervallet $I = [0, 2]$ delas in i två lika stora delintervall. (3p)

3. Bestäm Fourierserien till 2π-periodiska funktionen $f(x) = |x|, \ -\pi \le x \le \pi, \ \ p = 2\pi.$ (5p)

4. Betrakta denna ODE

$$\begin{cases} \dot{u}(t) + u(t) = f(t), & 0 < t \le T, \\ u(0) = u_0 \end{cases}$$

Använd likformig partition $0 = t_0 < t_1 < \cdots < t_n = T$ med steglängd k.

(a) Formulera Backward-Euler metoden (implicit-Euler metoden). (3p)

(a) Formulera Crank-Nicolson metoden. (3p)

5. Betrakta den homogena värmeledningsekvationen (10p)

$$\begin{cases} \dot{u}(x,t) - u''(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \ u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = 2x & 0 < x < 1. \end{cases}$$

Använd variabelseparationsmetoden för att bestämma $u(x,t)$.

6. Antag $a \in \mathbb{R}$. Visa att följande Laplacetransformer gäller: (6p)

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2} \quad , \qquad \mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}.$$

(OBS! Du kan använd $e^{iat} = \cos(at) + i \sin(at)$)

7. Betrakta begynnelsevärdesproblemet (BVP)

$$\begin{cases} -u'' + u = 2, & 0 < x < 1, \\ u'(0) = 0, \ \ u(1) = 0. \end{cases}$$

(a) Härled *variationsformulering.* (3p)

(b) Härled cG(1) finita element formulering (kontinuerliga styckvis linjära polynomer). Härled det linjära ekvationssystemet på formen $S\xi + M\xi = F$. Beräkna styvhetsmatrisen S (Stiffness matrix) och lastvektorn F (Load vector). Beräkna **ej** massmatrisen M (Mass matrix). (7p)

(OBS! Använd likformig partition med steglängd h och $\mathcal{T}_h: \ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1.$)

LYCKA TILL! \FS

Table of Laplace Transforms and trigonomerty

$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{at} f(t)$	$F(s-a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cos at$	$\frac{s}{s^2 + a^2}$
$\sin at$	$\frac{a}{s^2 + a^2}$
$\frac{t}{2a} \sin at$	$\frac{s}{(s^2 + a^2)^2}$
$\frac{1}{2a^3}(\sin at - at\cos at)$	$\frac{1}{(s^2 + a^2)^2}$
$2\sin a \sin b \; = \cos(a-b) - \cos(a+b)$	
$2\sin a \cos b \; = \sin(a-b) + \sin(a+b)$	
$2\cos a \cos b \; = \cos(a-b) + \cos(a+b)$	

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1. Take the Laplace transform

$$\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-3t}\}$$

$$\implies sY(s) - \underbrace{y(0)}_{=0} + 2Y(s) = \frac{1}{s+3}$$

$$\implies (s+2)Y(s) = \frac{1}{s+3} \qquad \implies \quad Y(s) = \frac{1}{(s+2)(s+3)}$$

Now we split the term on the right side:

$$Y(s) = \frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{A(s+3)+B(s+2)}{(s+2)(s+3)} = \frac{(A+B)s + 3A + 2B}{(s+2)(s+3)}$$

$$\implies \begin{cases} A + \ B = 0 \\ 3A + 2B = 1 \end{cases} \implies \quad A = 1, \quad B = -1$$

Therefore

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+3} = \frac{1}{s-(-2)} - \frac{1}{s-(-3)} \qquad \implies \quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-2t} - e^{-3t}$$

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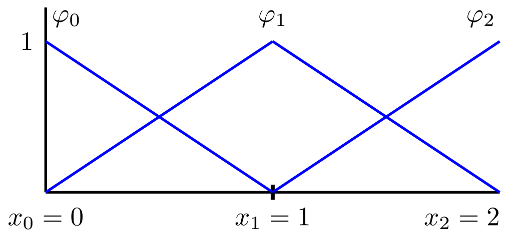
2. (a) Consider a linear combination of $p_1(t) = (t-1)^2,\ p_2(t) = (t-2)^2,\ p_3(t) = (t-3)^2$ as

$$\alpha_1(t-1)^2 + \alpha_2(t-2)^2 + \alpha_3(t-3)^2 = 0$$

Then for $t = 1, 2, 3$ we have

$$\implies \begin{cases} \alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 + \ \alpha_3 = 0 \\ 4\alpha_1 + \alpha_2 = 0 \end{cases} \implies \alpha_1 = \alpha_2 = \alpha_3 = 0$$

(b) $f(x) = 2x^2 - x$ and $x_0 = 0,\ x_1 = 1,\ x_2 = 2$ and $h = \frac{2-0}{2} = 1$.



$$\begin{aligned}
\pi_h f(x) &= \sum_{j=0}^{2} f(x_j)\varphi_j(x) = f(x_0)\varphi_0(x) + f(x_1)\varphi_1(x) + f(x_2)\varphi_2(x) \\
&= f(0)\varphi_0(x) + f(1)\varphi_1(x) + f(2)\varphi_2(x) \\
&= \varphi_1(x) + 6\varphi_2(x) \\
&= \begin{cases} \frac{x-x_0}{h} = x - 0 & x_0 \le x \le x_1 \\ \frac{x-x_2}{-h} + 6\frac{x-x_1}{h} = -(x-2) + 6(x-1) & x_1 \le x \le x_2 \end{cases} \\
&= \begin{cases} x & 0 \le x \le 1 \\ 5x - 4 & 1 \le x \le 2 \end{cases}
\end{aligned}$$

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3. f is an even function, so

$$b_n = 0, \quad n = 1, 2, \dots$$

And

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx = \frac{2}{\pi}\int_0^{\pi} x\ dx = \frac{2}{\pi}\Big[\frac{x^2}{2}\Big]_0^{\pi} = \pi$$

For $n = 1, 2, \dots,$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\ dx \\ &= \{u = x, \quad dv = \cos nx\ dx\} \\ &= \frac{2}{\pi}\underbrace{\Big[\frac{1}{n}x\sin nx\Big]_0^{\pi}}_{=0} - \frac{2}{n\pi}\int_0^{\pi}\sin nx\ dx = \frac{2}{n^2\pi}\big[\cos nx\big]_0^{\pi} = \frac{2}{n^2\pi}\big[(-1)^n - 1\big] \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\big((-1)^n - 1\big)\cos nx = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx \\ &= \frac{\pi}{2} - \frac{4}{\pi}\Big(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\Big) \end{aligned}$$

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4. (a) We stay at $t = t_l$:

$$\dot{u}(t_l) + u(t_l) = f(t_l) \quad\Longrightarrow\quad \frac{u(t_l) - u(t_{l-1})}{k} + u(t_l) \approx f(t_l)$$

Denoting $U \approx u$ and $U_l \approx u(t_l)$, the backward Euler method (implicit-Euler) is:

$$\frac{U_l - U_{l-1}}{k} + U_l = f(t_l) \quad\Longrightarrow\quad (1 + k)U_l = U_{l-1} + kf(t_l)$$

Hence

$$\begin{cases} U_0 = u_0 \\ (1 + k)U_l = U_{l-1} + kf(t_l), \qquad l = 1, 2, \dots, n \end{cases}$$

(b) We stay at $t = t_{l-\frac{1}{2}}$:

$$\dot{u}(t_{l-\frac{1}{2}}) + u(t_{l-\frac{1}{2}}) = f(t_{l-\frac{1}{2}}) \quad\Longrightarrow\quad \frac{u(t_l) - u(t_{l-1})}{k} + \frac{u(t_{l-1}) + u(t_l)}{2} \approx f(t_{l-\frac{1}{2}})$$

Denoting $U \approx u$ and $U_l \approx u(t_l)$, the Crank-Nicolson method is:

$$\frac{U_l - U_{l-1}}{k} + \frac{U_{l-1} + U_l}{2} = f(t_{l-\frac{1}{2}}) \quad\Longrightarrow\quad (1 + \frac{1}{2}k)U_l = (1 - \frac{1}{2}k)U_{l-1} + kf(t_{l-\frac{1}{2}})$$

Hence

$$\begin{cases} U_0 = u_0 \\ (1 + \frac{1}{2}k)U_l = (1 - \frac{1}{2}k)U_{l-1} + kf(t_{l-\frac{1}{2}}), \qquad l = 1, 2, \dots, n \end{cases}$$

Note: It is also correct, if you use

$$f(t_{l-\frac{1}{2}}) \approx \frac{f(t_{l-1}) + f(t_l)}{2}$$

then we have

$$\begin{cases} U_0 = u_0 \\ (1 + \frac{1}{2}k)U_l = (1 - \frac{1}{2}k)U_{l-1} + k\frac{f(t_{l-1}) + f(t_l)}{2}, \qquad l = 1, 2, \dots, n \end{cases}$$

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5. We look for the solution of the form

$$u(x,t) = X(x)T(t)$$

$$\overset{DE}{\Longrightarrow} \quad X\dot{T} - X''T = 0 \quad \overset{Divide\ by XT}{\Longrightarrow} \quad \frac{\dot{T}}{T}(t) = \frac{X''}{X}(x) = \lambda \quad \overset{\lambda=-\mu^2}{\Longrightarrow} \quad \begin{cases} X''(x) = -\mu^2 X(x) \\ \dot{T}(t) = -\mu^2 T(t) \end{cases} \qquad (*)$$

The first ODE in (*):

$$\begin{cases} X''(x) = -\mu^2 X(x) \\ X(0) = X(1) = 0 \end{cases} \Longrightarrow \begin{cases} X(x) = A\cos\mu x + B\sin\mu x \\ X(0) = X(1) = 0 \end{cases}$$

$X(0) = 0 \ \Rightarrow \ A = 0 \ \Rightarrow \ X(x) = B\sin\mu x$

$X(1) = 0 \ \Rightarrow \ B\sin\mu = 0 \ \overset{B\neq 0}{\Longrightarrow} \ \sin\mu = 0 \ \Rightarrow \ \mu = n\pi, \ n = 1, 2, \cdots$

$$\overset{B=1}{\Longrightarrow} \quad X_n(x) = \sin n\pi x, \quad n = 1, 2, \ldots.$$

The second ODE in (*) with $\mu = n\pi$:

$$\dot{T}(t) = -n^2\pi^2 T(t) \ \Longrightarrow \ T_n(t) = C_n e^{-n^2\pi^2 t}, \quad n = 1, 2, \ldots.$$

So for $\quad n = 1, 2, \ldots,$

$$u_n(x,t) = X_n(x)\ T_n(t) = C_n e^{-n^2\pi^2 t}\sin n\pi x$$

is a solution. By superposition principle, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t}\sin n\pi x$$

Use IC $u(x,0) = 2x$ (to compute C_n):

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin n\pi x = 2x$$

that is C_n is the coefficient of the Fourier sine series of $2x$:

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L 2x\sin n\pi x\ dx \overset{L=1}{=} 4\int_0^1 x\sin n\pi x\ dx = \{u = x, \quad dv = \sin n\pi x\ dx\} \\ &= -\frac{4}{n\pi}\underbrace{\big[x\cos n\pi x\big]_0^1}_{=(-1)^n} + \frac{4}{n\pi}\int_0^1 \cos n\pi x\ dx = \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^2\pi^2}\underbrace{\big[\sin n\pi x\big]_0^1}_{=0} \\ &= \frac{4}{n\pi}(-1)^{n+1} \end{aligned}$$

Hence

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2 t}\sin n\pi x$$

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6. See the lecture note (“L13_L14_L15.pdf”, page 4).

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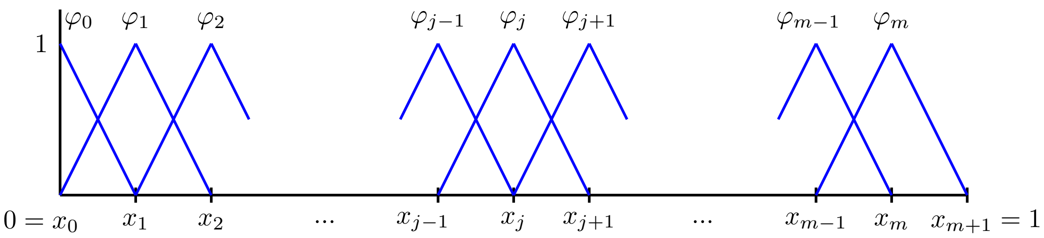
7. (a) Note that:

(1) there is no non-homogeneous Dirichlet BC, so we have only one function space,
(2) test function is zero at Dirichlet boundary points.

Define
$$V = \{v \mid v, v' \in L_2(0,1),\ v(1) = 0\}.$$
Multiply the DE by a test function $v \in V$, then integrate over $(0,1)$ and integrate by parts:
$$-\int_0^1 u''v\ dx + \int_0^1 uv\ dx = \int_0^1 2v\ dx$$
$$\Longrightarrow\ -u'(1)\underbrace{v(1)}_{=0} + \underbrace{u'(0)}_{=0}v(0) + \int_0^1 u'v'\ dx + \int_0^1 uv\ dx = 2\int_0^1 v\ dx$$
VF (variational form) is:

$$\text{Find } u \in V,\ \text{such that}$$
$$\int_0^1 u'v'\ dx + \int_0^1 uv\ dx = 2\int_0^1 v\ dx, \qquad \forall v \in V$$

(b) Consider a uniform partition with constant mesh size h:
$$\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$
The finite element space is
$$V_h = \{v \mid v \text{ is continuous p.w. linear on } \mathcal{T}_h,\ v(1) = 0\} = span\{\varphi_0, \varphi_1, \ldots, \varphi_m\}$$



FEM (finite element method) is:

$$\text{Find } U \in V_h,\ \text{such that}$$
$$\int_0^1 U'\chi'\ dx + \int_0^1 U\chi\ dx = 2\int_0^1 \chi\ dx, \qquad \forall \chi \in V_h$$

To write the matrix form:
choose $\quad \chi = \varphi_i, \quad n = 0, 1, \ldots, m$
substitute $\quad U(x) = \sum_{j=0}^m \xi_j \varphi_j(x)$

$$\int_0^1 \Big(\sum_{j=0}^m \xi_j \varphi_j'(x)\Big)\varphi_i'(x)\ dx + \int_0^1 \Big(\sum_{j=0}^m \xi_j \varphi_j(x)\Big)\varphi_i(x)\ dx$$
$$= 2\int_0^1 \varphi_i(x)\ dx, \qquad i = 0, 1, \ldots, m$$
$$\Longrightarrow \sum_{j=0}^m \underbrace{\Big(\int_0^1 \varphi_j'(x)\varphi_i'(x)\ dx\Big)}_{=S_{i,j}}\xi_j + \sum_{j=0}^m \underbrace{\Big(\int_0^1 \varphi_j(x)\varphi_i(x)\ dx\Big)}_{=M_{i,j}}\xi_j$$
$$= \underbrace{2\int_0^1 \varphi_i(x)\ dx,}_{=F_i} \qquad i = 0, 1, \ldots, m$$

that is the linear system of equations (with $m+1$ unknowns $\xi_0, \xi_1, \ldots, \xi_m$),

$$S\xi + M\xi = F$$

For the stiffness matrix we have (note that φ_0 is a half hat function):

<u>Stiffness matrix $S_{(m+1)\times(m+1)}$</u>:

$$S_{0,0} = \int_0^1 \varphi_0' \varphi_0' \; dx = \int_{x_0}^{x_1} \big(-\frac{1}{h}\big)^2 \; dx = \frac{1}{h}$$

For $i = 1, \ldots, m$,

$$S_{i,i} = \int_0^1 \varphi_i' \varphi_i' \; dx = \int_{x_{i-1}}^{x_i} \big(\frac{1}{h}\big)^2 \; dx + \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)^2 \; dx = \frac{2}{h}$$

For $i = 0, \ldots, m-1$,

$$S_{i,i+1} = S_{i+1,i} = \int_0^1 \varphi_i' \varphi_{i+1}' \; dx = \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)\big(\frac{1}{h}\big) \; dx = -\frac{1}{h}$$

That is

$$S = \frac{1}{h} \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(m+1)\times(m+1)}$$

<u>Load vector $F_{(m+1)\times 1}$</u>:

$$F_0 = 2\int_0^1 \varphi_0 \; dx = 2\int_{x_0}^{x_1} \varphi_0 \; dx = h$$

For $i = 1, \ldots, m$,

$$F_i = 2\int_0^1 \varphi_i \; dx = 2\int_{x_{i-1}}^{x_{i+1}} \varphi_i \; dx = 2h$$

Hence

$$F = h \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \end{bmatrix}_{(m+1)\times 1}$$