Exam 07.09.21
TMA683

(1) IC $\rightsquigarrow$ to determine a unique solution (0.5)

2 IC since second order DE (needs to integrate twice) (0.5)

(2) Yes! Obs $3x^2 = \underset{\uparrow \text{constant}}{3} \cdot x^2$ and we know from the lecture that $\{1, x, x^2\}$ is a basis. (1)

(3) Yes! $\langle f, f \rangle = \int_0^1 \underbrace{f^2(x)}_{\geq 0} dx + \int_0^1 \underbrace{(f'(x))^2}_{\geq 0} dx \geq 0$ OK (0.5)

$$\langle f, f \rangle = 0 \Leftrightarrow \int_0^1 f^2(x) dx = 0 \text{ and } \int_0^1 (f'(x))^2 dx = 0$$

But f and f' are continuous hence this can only work if $f = f' = 0$. (0.5)

(4) IVP on $[0, k]$ $\Leftrightarrow$ $y(k) \underset{\uparrow}{=} y_0 + \int_0^k f(y(t)) dt \underset{\uparrow}{\approx} y_0 + f(y_0) k \underset{\uparrow}{=} y_1$

Fundamental theorem (1) rectangle (2) This is Euler scheme! (1)

(5) No assumptions on specific values of u/v at $x=0, x=1 \rightsquigarrow$

test space (1) = trial space $= V = \{ v: [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \} = H^1(0,1)$. (1)

To get VF:

$$\int_0^1 f(x) v(x) dx = -\int_0^1 (a(x) u'(x))' v(x) dx \underset{\text{by part}}{=} \int_0^1 a(x) u'(x) v'(x) dx - a(1) \underbrace{u'(1)}_{u(1) - g_1} v(1) - a(0) \underbrace{u'(0)}_{u(0) - g_0} v(0)$$

VF reads: Find $u \in V$ s.t.

(1) $$\int_0^1 a(x) u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx + (u(1) - g_1) v(1) + (u(0) - g_0) v(0) \quad \forall v \in V$$

(6) a) Homogeneous Dirichlet BC at $x=0$ (1) $\Rightarrow V_h = \{ v : [0,1] \to \mathbb{R} : v$ cont. p.w. linear on partition with $v(0) = 0 \}$. (1)

b) FE problem reads: Find $u_h \in V_h$ s.t.

$$\int_0^1 (1+x) u_h'(x) \chi'(x) dx = (1+1) \cdot 1 \cdot \chi(1) \quad \forall \chi \in V_h$$

Stiffness matrix $S_{ij} = \left(\left((1+x) \varphi_j'(x), \varphi_i'(x) \right)_{L^2} \right)_{i,j=1}^3$ (1)

where $S'_{11} = \int_0^1 (1+x)\left(\varphi_1'(x)\right)^2 dx = \int_0^{1/3} (1+x)\frac{1}{h^2} + \int_{1/3}^{2/3} (1+x)\frac{1}{h^2}\,dx = 8$ //

$S'_{12} = \int_0^1 (1+x)\varphi_1'(x)\varphi_2'(x)\,dx = -\int_{1/3}^{2/3} (1+x)\frac{1}{h^2}\,dx = -9/2$ // ①

$S'_{13} = \int_0^1 (1+x)\varphi_1'(x)\varphi_3'(x)\,dx = 0$ // (disjoint)

c) Load vector $b_j = (1+1)\cdot\varphi_j(1)$ for $j=1,2,3$ ①. Hence $b = \begin{pmatrix}0\\0\\2\end{pmatrix}$ ①

"$\varphi_i(x_j) = \delta_{ij}$"

⑦ $\int_0^1 u'u'\,dx + \kappa' u(1)^2 + \kappa u(0)^2 = \int_0^1 f u\,dx \overset{C-S}{\leq} \left(\int_0^1 f^2 dx\right)^{1/2}\cdot\left(\int_0^1 u^2 dx\right)^{1/2} \leq$ ①

Hint $\leq C\cdot\left(\int_0^1 f^2 dx\right)^{1/2}\cdot\left(\int_0^1 (u')^2 dx + \underbrace{\kappa' u(1)^2}_{\geq 0} + \underbrace{\kappa u(0)^2}_{\geq 0}\right)^{1/2}$ ①

Hence $\left(\int_0^1 (u'(x))^2 dx + \kappa' u(1)^2 + \kappa u(0)^2\right)^{1/2} \leq C\left(\int_0^1 f^2(x)\,dx\right)^{1/2}$ ② Ok.

⑧ $u = v + iw$ into PDE: $iv_t - w_t - v_{xx} - iw_{xx} = 0$ or $\begin{cases} v_t - w_{xx} = 0 \\ -w_t - v_{xx} = 0 \end{cases}$

$\Leftrightarrow \begin{cases} v_t = w_{xx} \\ w_t = -v_{xx} \end{cases}$ $\cdot v$ and $\int_0^1 \dots dx$ / $\cdot w$ " " " $\Rightarrow$ $\int_0^1 v_t v\,dx = \int_0^1 w_{xx} v\,dx = -\int_0^1 w_x v_x\,dx$ ①

$\int_0^1 w_t w\,dx = -\int_0^1 v_{xx} w\,dx = \int_0^1 v_x w_x\,dx$ ①

Add and get $\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx = 0$ or $\frac{1}{2}\frac{d}{dt}\|u\|^2 = 0$ ①

Hence $\|u(\ast,t)\|_{L^2} = \|u_0\|_{L^2}$ $\forall t$ $\quad |u|^2 = v^2 + w^2$ ①

⑨

$\mathcal{L}\{e^{3t}\}(s) = \int_0^\infty e^{3t}\cdot e^{-st}\,dt = \int_0^\infty e^{(3-s)t}\,dt = \lim_{L\to\infty} \left.\frac{e^{-(s-3)t}}{-(s-3)}\right|_{t=0}^{L} =$ ①

$= \frac{1}{s-3}$ for $s > 3$. ①

Convergent for $s-3>0$ or $s>3$

⑩ Table ① $G(s) = \mathcal{L}\{g(t)\}(s) = \frac{s-9}{(s-9)^2+49}$ (shift property) ①

(11)

$Y(s) := \mathcal{L}\{y\}(s)$

• Take LT: $\mathcal{L}\{y''\}(s) - 6\mathcal{L}\{y'\}(s) + 9\mathcal{L}\{y\}(s) = 0 \iff$

$(s^2 - 6s + 9)Y(s) = s - 5$ (1) or $Y(s) = \frac{s-5}{(s-3)^2}$ (1) $= \frac{s-3}{(s-3)^2} + \frac{3}{(s-3)^2}$ (1) $- \frac{5}{(s-3)^2}$

• Take ILT: $y(t) = e^{3t} - 2te^{3t}$ (1)

(12) Let $m \neq n$ integers;

$(\sin(mx), \sin(nx)) = \int_0^{\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^{\pi} (\cos((m-n)x) - \cos((m+n)x))\,dx$

$= 0$ (1) $\Rightarrow$ orthogonal set. (1)

This property is important as it is used to find the coefficient of the Fourier sine series of a function, see lecture. (1)

(13) f is even $\Rightarrow$ $b_n = 0$ $\forall n$ (1)

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = 2\int_0^{\pi}\cos(nx)\,dx - \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx =$

$= \begin{cases} \frac{2}{\pi n^2}(1-(-1)^n) & n \neq 0 \quad (1) \\ \pi & n = 0 \quad (1) \end{cases}$

Hence $f(x) \sim \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k+1)^2}\cos((2k+1)x)$ (1)

(14) a) $f(x)$ odd $\Rightarrow$ $a_0 = a_n = 0$ (1) and $b_n = \frac{2}{n}(-1)^{n+1}$ (1)

Hence $x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$. (1)

b) Parseval: $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}\right)^2$ (2) $\iff$

$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\frac{2\pi^3}{3}$ (1) or $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ (1)

(15) Separation $u(x,t) = X(x)T(t)$ into PDE

constant

(i) $X(x)T'(t) + X''(x)T(t) + \frac{1}{x^2}X(x)T(t) = 0$ or (i) $\frac{T'(t)}{T(t)} = -\frac{X''(x)}{X(x)} - \frac{1}{x^2}\frac{X(x)}{X(x)} = -\lambda$

(i) $T'(t) = T(t)(-\lambda)$ (1)

$-X''(x) - \frac{1}{x^2}X(x) + \lambda X(x) = 0$ (1)

16 a) Since u_1 and u_2 solve the given PDE, using superposition principle, one gets that $V = u_1 - u_2$ solves

$$\begin{cases} V_{tt} - V_{xx} = 0+0=0 \\ V(0,t) = V(1,t) = 0+0=0 \\ V(x,0) = f(x) - f(x) = 0, \quad V_t(x,0) = 0. \end{cases}$$

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b) Conservation of energy gives

$$\|V_t(\cdot,t)\|^2_{L^2(0,1)} + \|V_x(\cdot,t)\|^2_{L^2(0,1)} = \|V_t(\cdot,0)\|^2_{L^2(0,1)} + \|V_x(\cdot,0)\|^2_{L^2(0,1)} =$$

$$\underset{\uparrow}{=} 0+0=0$$

Def initial values for V

Hence $\|V_t(\cdot,t)\|_{L^2} = \|V_x(\cdot,t)\|_{L^2} = 0$ $\forall t$ on, since V is continuous,

Def V

$V_t(x,t) = V_x(x,t) = 0$ $\forall x,t$ on $V(x,t) = \text{Const} \underset{\substack{\uparrow \\ V(x,0)=0}}{=} 0 \Rightarrow u_1 = u_2$ //

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