Examination, 7 April 2021
TMA683

Read this before you start!
Aid: Anything but collaboration.
A table of Laplace transforms can be found at the end of the exam.
If something is not clear you can ask to talk to me over zoom.
In case of problems, I'll try to contact all of you via the course canvas module.
Read all questions first and start to answer the ones you like most.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for someone to find the correct one. Justify your answers!
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Write your cid or first numbers of your personnummer.
Use a proper pen, order your answers, use an app like camscanner or equivalent, and check your final scan before uploading it.
The test has 5 pages and a total of 50 points.
Valid bonus points will be added to the total score if needed.
You will be informed when the exams are corrected.
"Jag försäkrar att jag gjort tentan på egen hand utan att få hjälp från någon annan person och att jag själv formulerat alla lösningar."
Check the box □
Good luck!

Some exercises were taken from, or inspired by, materials from *Jim Lambers*, *Mats Larson*, *Pierre Maréchal*, *Hamid Meziani*, *Axel Målquist*, *www.projectrhea.org*, *www.math24.net*.

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1. Consider the differential equation

$$\ddot{x}(t) = \cos(x).$$

Why does one add initial condition(s) to this problem? How many initial condition(s) does one add to the above problem? (1p)

2. Is the set $\{3x^2, 1, x\}$ a basis of the set of all polynomials of degree two or less? (1p)

3. Let V be the space of continuously differentiable (real valued) functions on $[0,1]$. Is the bilinear map (defined for $f, g \in V$)

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x + \int_0^1 f'(x)g'(x)\,\mathrm{d}x$$

positive-definite (i. e. $\langle f, f\rangle \geq 0$ with equality if and only if $f \equiv 0$, the zero function)? (1p)

4. Let $k > 0$. Apply a rectangle rule (area approximated by a rectangle of height $f(y_0)$ and length k) to the integral form of the IVP

$$\begin{cases} \dot{y}(t) = f(y(t)) \\ y(0) = y_0 \end{cases}$$

for $t \in [0, k]$. From this last computation, using $y_1 \approx y(k)$, derive the forward Euler scheme for solving the above IVP. (4p)

Hint: If you are not able to find the integral equation, consider the following

$$y(k) = y_0 + \int_0^k f(y(s))\,ds$$

to answer the rest of the exercise. Points will be deduced accordingly.

5. Let $\Omega = [0, 1]$, $a\colon \Omega \to (0, \infty)$, $g_0, g_1 \in \mathbb{R}$, and $f\colon \Omega \to \mathbb{R}$ nice. Consider the following problem (for $x \in (0, 1)$)

$$\begin{cases} -(a(x)u'(x))' = f(x) \\ a(0)u'(0) = u(0) - g_0, \\ a(1)u'(1) = u(1) - g_1. \end{cases}$$

Which test and trial spaces would you consider to find a variational formulation of the above BVP? Give this variational formulation. Justify your answers. (3p)

6. Consider the following BVP

$$\begin{cases} -((1 + x)u'(x))' = 0 \\ u(0) = 0, \quad u'(1) = 1, \end{cases}$$

where $x \in (0, 1)$. Assume that the interval $[0, 1]$ is divided into three subintervals of equal length $h = 1/3$.

(a) Define the corresponding finite element space V_h. (2p)

(b) Compute the entries of the first line of the stiffness matrix corresponding to this problem. (2p)

(c) Compute all entries of the load vector corresponding to this problem. (2p)

7. Let $\kappa \geq 0$ and $f\colon [0, 1] \to \mathbb{R}$ nice. Consider the variational formulation: Find $u \in H^1(0, 1)$ such that

$$\int_0^1 f(x)v(x)\,\mathrm{d}x = \int_0^1 u'(x)v'(x)\,\mathrm{d}x + \kappa u(1)v(1) + \kappa u(0)v(0)$$

for all $v \in H^1(0, 1)$. Take $v = u$ above and show that

$$\int_0^1 |u'(x)|^2\,\mathrm{d}x + \kappa|u(1)|^2 + \kappa|u(0)|^2 \leq C \int_0^1 |f(x)|^2\,\mathrm{d}x$$

for some $C > 0$. You may use the estimate $\int_0^1 |u(x)|^2 \,\mathrm{d}x \leq C' \int_0^1 |u'(x)|^2 \,\mathrm{d}x$ for some constant $C' > 0$. (4p)

8. Let two initial values v_0, w_0. Consider the system of partial differential equations (for $x \in [0,1]$ and $t \in [0,T]$)

$$\begin{cases} v_t(x,t) - w_{xx}(x,t) = 0 \\ -w_t(x,t) - v_{xx}(x,t) = 0 \\ v(x,0) = v_0(x), w(x,0) = w_0(x) \end{cases}$$

with homogeneous Dirichlet boundary conditions (for both functions).
Show that the quantity $\|v(\cdot,t)\|^2_{L^2(0,1)} + \|w(\cdot,t)\|^2_{L^2(0,1)}$ is conserved for all time $t > 0$. (4p)
Hint: Multiply the equations for v and for w with two clever choices of functions, integrate over the spatial domain and work out the rest. Try to inspire yourself with what we did for the linear wave equation in the lecture.

9. Using only the definition of the Laplace transform, compute

$$\mathcal{L}\{\mathrm{e}^{3t}\}(s)$$

and give the domain of definition of the above function. (2p)

10. Find the Laplace transform of the following function

$$g(t) = \mathrm{e}^{9t}\cos(7t). \qquad (2p)$$

11. Use Laplace transforms to solve the following initial value problem

$$\begin{cases} y''(t) - 6y'(t) + 9y(t) = 0 \\ y(0) = 1 \\ y'(0) = 1. \end{cases} \qquad (4p)$$

12. Consider the set of functions $\{\sin(nx)\}_{n=1}^{\infty}$ with the inner product

$$\langle f, g \rangle = \int_0^{\pi} f(x)g(x)\,\mathrm{d}x.$$

Is this set orthogonal? Why is this set important (in the context of this course)? (3p)

13. Compute the Fourier series of the 2π-periodic function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \pi - |x| \quad \text{for} \quad -\pi < x \leq \pi. \qquad (4p)$$

14. Consider the function $f(x) = x$ for $x \in [-\pi, \pi]$.

    (a) Compute the Fourier series expansion of f. (3p)

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (4p)

Hint: If you are not able to do the first part, you may use

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx).$$

Points will be deduced accordingly.

15. From the partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{x^2} u(x,t) = 0$$

find one differential equation involving only x and one involving only t. (2p)

16. Consider the wave equation $(0 < x < 1, 0 < t \leq T)$

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

with given (nice) functions f, g. The goal is now to show uniqueness of a continuously differentiable solution to the above problem.

(a) Assume that u_1 and u_2 are two continuous solutions to the above PDE and set $v = u_1 - u_2$. Use the superposition principle to show that the function v satisfy the partial differential equation

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) = 0 \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = 0, v_t(x,0) = 0. \end{cases}$$

(2p)

(b) Using the conservation of energy property, deduce that $v(x,t) = 0$ for all $x \in [0,1]$ and $t \in [0,T]$. This hence shows that $u_1 = u_2$. (2p)

Table of Laplace Transforms and trigonometry

$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\mathrm{e}^{-at} f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$\mathrm{e}^{-Ts}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin(bt) - bt\cos(bt))$	$\frac{1}{(s^2+b^2)^2}$

$$
\begin{aligned}
2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b)\\
2\sin(a)\cos(b) &= \sin(a-b) + \sin(a+b)\\
2\cos(a)\cos(b) &= \cos(a-b) + \cos(a+b)
\end{aligned}
$$