Tenta TMA683, 16.01.21

① $\int_{-\pi}^{\pi} f(x) f'(x)\,dx \underset{\text{by parts}}{=} \underbrace{f(x)f(x)\Big|_{-\pi}^{\pi}}_{=0 \text{ since } f \text{ } 2\pi\text{-periodic}} - \int_{-\pi}^{\pi} f'(x) f(x)\,dx \Rightarrow \int_{-\pi}^{\pi} f'(x) f(x)\,dx = 0 \Rightarrow f' \perp f$

② a) $\dot{y}(t) = f(y(t)),\ y(0) = y_0 \underset{\text{fund. theorem calculus}}{\Longleftrightarrow} y(t) = y_0 + \int_0^t f(y(s))\,ds$

b) $t = k \Rightarrow y(k) = y_0 + \int_0^k f(y(s))\,ds \underset{\text{Trapezoidal rule}}{\approx} y_0 + \frac{k}{2}\left(f(y_0) + f(y(k))\right)$

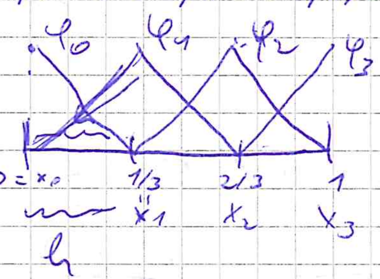
c) $y_1 = y_0 + \frac{k}{2}\left(f(y_0) + f(y_1)\right) \approx y(k)$

C-N!

③ See lecture, hom. Dirichlet BC $\Rightarrow$ test space = trial space = $\{ v : [0,1] \to \mathbb{R} ;\ v, v' \in L^2(0,1) \text{ and } v(0) = v(1) = 0 \}$

The sol. to this BVP lives in the test/trial space.

④ a) See below + lecture: $V_h = \text{span}(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$



b) FE problem: Find $u_h \in V_h$ s.t. $\int_0^1 u_h'(t) \chi'(x)\,dx = 3\chi(1) \quad \forall \chi \in V_h$

or $2\int_0^1 \varphi_0'(x)\varphi_j'(x)\,dx + \sum_{i=1}^{3} \xi_i \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx = 3\varphi_j(1)$ for $j = 0,1,2,3$.

1st. entry load vector: $3\varphi_0(1) - 2\int_0^1 \varphi_0'(x)\varphi_0'(x)\,dx = 0 - 2\int_0^1 \frac{1}{h^2}\,dx = -2h^{-1}$ ok,

last entry matrix : $\int_0^1 \varphi_3'(x)\varphi_3'(x)\,dx = \int_{2/3}^1 \frac{1}{h^2}\,dx = \frac{1}{h}$ ok.

last entry ~~matrix~~ load vector : $3\varphi_3(1) - 2\int_0^1 \varphi_0'(x)\varphi_3'(x)\,dx = 3 - 0 = 3$ ok

⑤ See chapter IV lectures; Error to BVP $\begin{cases} -u''(x) = f(x) \\ u(0) = u(1) = 0 \end{cases}$ for FE sol. reads $\|u - u_h\|_E \leq C \cdot h \cdot \|u''\|_{L^2(0,1)}$, where $\|v\|_E^2 = (v', v')_{L^2}$

⑥
a) $\dot{y}(t) = y(t)^2$, $y(0) = 1$, b) Euler scheme c) $h = \frac{1}{10}$ d) $y_1 = y_0 + h y_0^2 = 1 + \frac{1}{10} = \frac{11}{10}$

⑦ $u_t - u_{xx} = 0 \quad | \cdot u$ and $\int_0^1 dx \underset{\text{by parts}}{\Rightarrow} \frac{1}{2}\frac{d}{dt}\int_0^1 u(x,t)^2 dx \underbrace{-u_x u\Big|_{x=0}^{1}}_{=0 \text{ by BC.}} + \underbrace{\int_0^1 u_x(x,t)^2 dx}_{\geq 0} \underset{\text{Def } L^2\text{-norm}}{\Leftrightarrow} \frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(0,1)} \leq 0$

⑧ $\mathcal{L}\{1\}(s) \underset{\text{Def LT}}{=} \int_0^\infty 1 \cdot e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_{t=0}^{\infty} = \left(-\frac{1}{s}\right)\left(\lim_{t\to\infty} e^{-st} - 1\right) \underset{\text{Defined for } s>0}{=} \left(-\frac{1}{s}\right)(0-1) = \frac{1}{s}$ for $s > 0$.

⑨ $y(t) = \mathcal{L}^{-1}\{Y\}(t) \underset{\text{linearity}}{=} 6\,\mathcal{L}^{-1}\{\frac{1}{s+1}\}(t) + 7\,\mathcal{L}^{-1}\{\frac{1}{s^5}\}(t) \underset{\text{Table}}{=} 6e^{-t} + \frac{7}{4!}t^4$

⑩ a) Take LT : $sY(s) - 2 + 2Y(s) + \frac{5}{s}Y(s) = \frac{3}{s}$, where $Y(s) = \mathcal{L}\{y\}(s)$.

b) Find $Y(s) = \frac{2s+3}{(s+1)^2+4} = \frac{2(s+1)}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$

c) Take ILT : $y(t) \underset{\text{Table}}{=} e^{-t}\left(2\cos(2t) + \frac{1}{2}\sin(2t)\right)$

⑪ This is already a FS $\Rightarrow \frac{a_0}{2} = 2$, $b_1 = 6$ rest of F. coeff. $= 0$

(12) $2L = 4 \Rightarrow L = 2$ and $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$, where

$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx = \frac{1}{2}\int_0^4 x\,dx = 4$

$b_n = \frac{1}{2}\int_0^4 x \sin\left(\frac{n\pi x}{L}\right)dx = \ldots = -\frac{4}{n\pi}$ for $n = 1, 2, \ldots$

$a_n = \frac{1}{2}\int_0^4 x \cos\left(\frac{n\pi x}{L}\right) dx = \ldots = 0$ for $n = 1, 2, 3, \ldots$

$\Rightarrow f(x) \sim 2 - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2}\right)$

(13) Set $h = f - g$ and $c_n(h) = c_n(f) - c_n(g) = 0$ and obs. h is continuous

Fourier coeff. for h, f, g Hypothesis

Parseval $\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(t)|^2 dt = \sum_{n \in \mathbb{Z}} |c_n(h)|^2 = 0$ hence (Hint) $h \equiv 0$ hence $f \equiv g$.

(14) Set $u(x,t) = X(x)T(t)$ into PDE: $T''X + T'X + TX = TX''$ or $\frac{T''}{T} + \frac{T'}{T} + 1 = \frac{X''}{X} = \lambda$ or $\begin{cases} T'' + T' + T = \lambda T \\ X'' = \lambda X \end{cases}$

constant

(15) Superposition principle allows to write $u(x,t) = v(x) + w(x,t)$, where

v, w solves $\begin{cases} -v''(x) = 1 \\ v(0) = v(1) = 0 \end{cases}$ $\begin{cases} w_{tt} - w_{xx} = 0 \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = f(x) - v(x), \quad w_t(x,0) = g(x) \end{cases}$ (see lecture)

BVP PDE