A

TMA683: Exam 15.01.22

1 a) No. Unknown in ODE depends on 1 variable 1

b) $\mathcal{B}^{(2)}(\mathbb{R})$ 1

c) The FE sol! 1

d) Yes 1

e) LT is an integral 1

2 u and v lin. indep $\Rightarrow \exists\, a,b \in \mathbb{R}$ (not both zero) s.t. $au+bv=0$

hence $u = cv$ for some $c \in \mathbb{R}$. 1

LHS of C-S reads $|\langle u,v\rangle| = |\langle c\cdot v, v\rangle| = |c|\cdot|\langle v,v\rangle|$ 1

RHS of C-S reads $\|u\|\cdot\|v\| = \sqrt{c^2\langle v,v\rangle}\cdot\sqrt{\langle v,v\rangle} = |c|\cdot|\langle v,v\rangle|$ 1

ok!

using def + prop. of norm/inner prod.

3 $\text{Area} = \int_0^1 x^2\,dx \underset{\text{Def Trap.}}{\approx} \frac{1-0}{2}\cdot\left(x^2\Big|_{x=0} + x^2\Big|_{x=1}\right) = \frac{1}{2}(0+1) = \frac{1}{2}$ 0.5 0.5

4 Hom. Dirichlet BC $\Rightarrow$ Test = trial $= H_0^1(0,1)$ 1 1 1

The sol. to BVP belongs to $H_0^1(0,1)$. 1

5 Disjoint support $\Rightarrow$ Mass/stiffness matrices have a lot of zeros. 1 1

6 a) Integrate $\Rightarrow u(x) = c_0 + c_1 x$, BC $\Rightarrow c_0 = -2,\ c_1 = 1$ 1

Sol. to BVP reads $u(x) = -2+x$ 1

b) Multiply by test fct $v \in V$ and by parts.

$$\underbrace{-u'(2)\,\underset{=0}{v(2)}}_{\text{kill}} + \underbrace{\underset{=1}{u'(0)}\,v(0)}_{\text{keep}} + \int_0^2 u'(x)\,v'(x)\,dx = 0 \quad \forall v \in V$$

(VF) Find $u \in V$ s.t. $(u',v')_{L^2} = 0 - v(0) \quad \forall v \in V$, 1

$V = \{v \in H^1 \text{ s.t. } v(2) = 0\}$ 1

c) $0 = x_0 \underbrace{\rule{3em}{0.4pt}}_{h} x_1 = 2 \Rightarrow 1$ hat fct $\varphi_0(x)$ (1)

(FE) Find $u_h = \xi_0 \varphi_0 \in span(\varphi_0)$ s.t. $(u_h', \varphi_0')_{L^2} = -\varphi_0(0)$ (1)

d) That is $\xi_0(\varphi_0', \varphi_0')_{L^2} = -1$ or $\xi_0 = -2$ (1) $\Rightarrow u_h(x) = x - 2$ (1), i.e. exact sol.

(7) $\|u - u_h\|_E \leq C \cdot h$ (2)

(8) $y'(x) = e^x = y(x)$ and $y(0) = e^0 = 1$ ok (1)

Euler reads $y_{n+1} = y_n + h y_n$ (1)

$n=1 \to y_1 = \frac{3}{2} \approx y(\frac{1}{2})$, $n = 2 \leadsto y_2 = \frac{9}{4} \approx y(1) = e^1$ (1)

(9)

(FE) For each $t \in (0,T)$, find $u_h(\cdot,t) \in H_0^1(0,1)$ s.t.

$(u_{h,t}(\cdot,t), \chi) + (u_{h,x}(\cdot,t), \chi') + (u_h(\cdot,t), \chi) = (f, \chi) \quad \forall \chi \in H_0^1$ (1)

Write $u_h(x,t) = \sum_{j=1}^{m} \xi_j \varphi_j(x)$ and take $\chi = \varphi_i$, $i = 1, \dots, m$, to get

$M\dot{\xi}(t) + A\xi(t) + M\xi(t) = F(t)$, where $M = \left((\varphi_j, \varphi_i)\right)_{i,j=1}^{m}$ (1), $A = \left((\varphi_j', \varphi_i')\right)_{i,j=1}^{m}$ (1), $F = \left((f, \varphi_j)\right)_{j=1}^{m}$ (1)

(10) e^{t^2} not of exp. order $\Rightarrow$ LT $\nexists$

(11) Set $f(t) = \sin(bt) \to f'(t) = b\cos(bt)$ and $f(0) = 0$

$\Rightarrow \mathcal{L}\{f'(t)\}(s) = b\,\mathcal{L}\{\cos(bt)\}(s) \Rightarrow \mathcal{L}\{\cos(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f'(t)\}(s)$ (1) $= \frac{s}{b} \cdot \frac{b}{s^2+b^2} - 0 = \frac{s}{s^2+b^2}$. (1)

(12) (i) Take LT: $(s^2 + 5s + 6)Y(s) = \frac{1}{s} + 1$ or $Y(s) = \frac{s+1}{s(s+2)(s+3)}$ (1)

(ii) Part. frac. $Y(s) = \frac{1}{6s} + \frac{1}{2}\frac{1}{s+2} - \frac{2}{3}\frac{1}{s+3}$ (1)

(iii) Take ILT: $y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \dots = \frac{1}{6}\Theta(t) + \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t}$ (2)

(13) Orth. $\Leftrightarrow (f,g)_{L^2(-1,1)} = 0$ (1). Now, f even and g odd $\Rightarrow f \cdot g$ odd $\Rightarrow (f,g)_{L^2} = 0$ (1)

(14) Already a FS with $\frac{a_0}{2} = 2$ (1), $a_5 = 6$, reste $= 0$ (1)

(15) f odd $\Rightarrow a_n = 0$ (1)

$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(nx)\,dx$ (1) $+ \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$ (1) $= \frac{2}{n\pi}(1 - \cos(n\pi)) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$ (1)

(16) a) Take $x = \pi$ and get $\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$

or $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (0,5)

b) $\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{1}{2} \left(2 \frac{\pi^2}{3} \right)^2 + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \right)^2$ (1)

$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ (1)

(17) Ansatz $F(x,y) = X(x) \cdot Y(y)$ (1) gives $\frac{X'(x)}{2x X(x)} = -\frac{Y'(y)}{Y(y)} = \lambda$

$\Rightarrow$ $X'(x) = 2x\lambda X(x)$ (0,5)

$Y'(y) = -\lambda Y(y)$ (0,5)

$\sim$

U < 20p, 3: 20-29p, 4: 30-39p, 5: 40-50p