Recall: LT $\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$

ILT $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$ (Table)

Ex: $\mathcal{L}\{\theta(t)\}(s) = \frac{1}{s}$

$\mathcal{L}^{-1}\{\frac{1}{s}\}(t) = \theta(t)$

Properties: $\mathcal{L}\{f'(t)\}(s) = s\underbrace{F(s)}_{=\mathcal{L}\{f\}} - f(0)$ //

Method of partial fractions:

Prob: Compute $\mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \frac{Q(s)}{P(s)}$,

where P, Q are polynomials with

$\deg(Q) < \deg(P)$

Idea: Use partial fractions to decompose F

into smaller/easier blocks

We consider 3 typical examples

(i) P(s) is quadratic and has 2 distinct roots/zeros:

$$F(s) = \frac{Q(s)}{P(s)} = \frac{2s-8}{s^2-5s+6} = \frac{2s-8}{(s-2)(s-3)} =$$

two roots 2, 3

$$\stackrel{!}{=} \frac{A}{s-2} + \frac{B}{s-3}$$

simpler

Find A and B:

$$F(s) = \frac{2s-8}{(s-2)(s-3)} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)} = \frac{s(A+B)-3A-2B}{(s-2)(s-3)}$$

Compare coeff: s: $2 \stackrel{!}{=} A+B$

s^0: $-8 \stackrel{!}{=} -3A-2B$ $\Rightarrow \dots \Rightarrow$

$A=4$ and $B=-2$

$$\leadsto F(s) = \frac{4}{s-2} - \frac{2}{s-3}$$

Finally, ILT of F is given by

$$\mathcal{L}^{-1}\{F(s)\}(t) \underset{\text{linearity}}{=} 4\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - 2\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t)$$

$$\underset{\text{Table}}{=} 4\cdot e^{2t} - 2e^{3t} \quad //$$

iii) $P(s)$ is quadratic with one double root

$$F(s) = \frac{Q(s)}{P(s)} = \frac{s+1}{(s+2)^2}$$

root -2

$$F(s) = \frac{s+1}{(s+2)^2} \overset{!}{=} \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

double root

We find A and B:

$$\frac{s+1}{(s+2)^2} \overset{!}{=} \frac{A(s+2)+B}{(s+2)^2} = \frac{As + (2A+B)}{(s+2)^2}$$

Compare coeff: s^1: $1 \overset{!}{=} A \;\rightarrow\; A = 1$

s^0: $1 \overset{!}{=} 2A + B \qquad B = -1$

(L) $F(s) = \frac{1}{s+2} - \frac{1}{(s+2)^2}$

Finally, one gets

$$\mathcal{L}^{-1}\{F(s)\}(t) \underset{\text{linearity}}{=} \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}(t)$$

$$\underset{\text{table}}{=} e^{-2t} - e^{-2t} \cdot t$$

$\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t \right.$

$\left. + \text{ shifting prop.} \right)$

(iii) <u>$P(s)$ is a quadratic polyn. with two complex roots:</u>

$$F(s) = \frac{Q(s)}{P(s)} = \frac{s+1}{s^2+4s+5} \underset{\text{complete the square}}{=} \frac{s+1}{(s+2)^2+1} =$$

five

$$= \frac{s+1 +1 -1}{(s+2)^2+1} = \frac{s+\boxed{2}}{(s+\boxed{2})^2+1} - \frac{1}{(s+2)^2+1}$$

$$\underbrace{\frac{s+2}{(s+2)^2+1}}_{\frac{\hat{s}}{(\hat{s})^2+1}} \qquad \underbrace{\frac{1}{(s+2)^2+1}}_{\frac{1}{(\hat{s})^2+1}}$$

$\hat{s} = s+2 \rightsquigarrow$ Shifting

(4) Finally, using the table, one gets

$$\mathcal{L}^{-1}\{F(s)\}(t) = e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

shifting / formulas for $\cos(bt)$ and $\sin(bt)$

4) Applications of Laplace transform

<u>Goal</u>: Use LT solve particular ODE and integral equations

<u>Idea</u>: (i) Take LT of the problem to get an algebraic equation

(ii) Solve algebraic equation

(iii) Take ILT to get back the sol. to the original problem.

<u>Ex</u> (Solve ODE with LT)

<u>Prob</u>: $\begin{cases} y'(t) + 2y(t) = 12e^{3t} \\ y(0) = 3 \end{cases}$

(i) Take LT: $\mathcal{L}\{y'(t)\}(s) + 2\mathcal{L}\{y(t)\}(s) = 12 \underbrace{\mathcal{L}\{e^{3t}\}(s)}$

Set $Y(s) := \mathcal{L}\{y(t)\}(s)$ and the above gives:

$$sY(s) - \underset{\underset{3}{\shortparallel}}{y(0)} + 2Y(s) = 12\frac{1}{s-3} \quad \text{(Table)}$$

$$\Rightarrow (s+2)Y(s) = \frac{12}{s-3} + \frac{3(s-3)}{s-3} = \frac{3s+3}{s-3}$$

algebraic eq

(ii) $\Rightarrow Y(s) = \dfrac{3s+3}{(s-3)(s+2)}$

(iii) Take ILT to get back $y(t)$ (sol. ODE)

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\Big\{ \underbrace{\frac{3s+3}{(s-3)(s+2)}}_{\text{partial fractions}} \Big\}(t)$$

$$\frac{3}{5}\,\frac{1}{s-3} + \frac{12}{5}\,\frac{1}{s+2}$$

$$\Rightarrow \dots \Rightarrow \quad y(t) = \frac{3}{5}e^{+3t} + \frac{12}{5}e^{-2t}$$

⚠ Check if $y(t)$ satisfies the ODE

$$y(0) = 3$$

$$y'(t) + 2y(t) = 12e^{3t}$$

Ex: (Solve integral equation using LT)

Prob: Current $i(t)$ in electric circuit is given by

$$\begin{cases} i(t) + \int_0^t i(\tau)\,d\tau = v(t) \\ i(0) = 0, \end{cases}$$

where $v(t) = \theta(t-1) - \theta(t-2)$.

(i) Take LT and set $I(s) = \mathcal{L}\{i(t)\}(s)$;
and get:

$$I(s) + \mathcal{L}\left\{\int_0^t i(\tau)\,d\tau\right\}(s) = \mathcal{L}\{\theta(t-1)\}(s) - \mathcal{L}\{\theta(t-2)\}(s)$$

Using the table we get

$$\underbrace{I(s) + \frac{I(s)}{s}}_{I(s)\left(\frac{s+1}{s}\right)} = \underbrace{\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}}_{\frac{e^{-s} - e^{-2s}}{s}}$$

(ii) Find $I(s)$:

$$I(s) = \frac{e^{-s} - e^{-2s}}{s+1}$$

(iii) Take ILT to get $i(t)$:

$$i(t) = \mathcal{L}^{-1}\{I(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+1}\right\}(t)$$

Looking at the table, we get

$$i(t) = e^{-(t-1)}\,\theta(t-1) - e^{-(t-2)}\,\theta(t-2)$$

shifting + $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) = \theta(t)$

$$\hookrightarrow e^{-Ts}F(s) = f(t-T)\,\theta(t-T)$$