Examination, 11 April 2022
TMA683

Read this before you start!
I'll try to come at ca. 09:45. You can ask for calling me (0317723021) in case of additional questions.
Aid: Personal pocket calculator.
A table of Laplace transforms is provided at the end of the exam.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible. ***Thank you****.*
The test has 5 pages and a total of 50 points.
Preliminary grading limits: 3: 20-29p, 4: 30-39p and 5: 40-50p.
Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.

Good luck!

Some exercises were taken from, or inspired by, materials from *A. Bodin*, *P. Dawkins*, *A. Massing*.

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1. Provide concise answers to the following short questions:

   (a) Is the set $\{1, x, x^2\}$ a basis of the linear space of all polynomials of degree two or less? (1 p)

   (b) Which function lives in the trial space in the variational form of a PDE? (1 p)

   (c) Use one step of the midpoint rule to approximate the area under the function $f(x) = x^2$ between $x = 0$ and $x = 1$. (1 p)

   (d) Is the energy of the homogeneous linear wave equation with homogeneous Dirichlet BC a conserved quantity? (1 p)

   (e) Why is adaptivity useful in this course? (1 p)

2. Let $f\colon [0,1] \to \mathbb{R}$ be continuous and let n be a given integer. Consider distinct data points $(x_j, f(x_j))_{j=0}^{n}$ with $0 = x_0 < x_1 < \ldots < x_n = 1$.

   (a) Provide the definition of Lagrange polynomials. (1 p)

   (b) Provide the definition of a Lagrange interpolant $\pi_n(x)$ for the given data points. What is the degree of this polynomial interpolant? (2 p)
   Hint: If you have not done the previous item, then you can still provide this definition.

(c) Using only the definition of a Lagrange interpolant and a property of Lagrange polynomials, show that indeed $\pi_n(x_j) = f(x_j)$ for $j = 0, \dots, n$. (1 p)

3. Let $T = 2$. Consider the IVP (for $t \in [0, T]$)

$$\begin{cases} \dot{y}(t) = y(t)^2 \\ y(0) = 1. \end{cases}$$

We are interested in a numerical approximation of the solution to the above IVP by the explicit Euler scheme (with time step $h = 1/100$). Complete the following pseudo code by providing the missing terms WW,XX,YY,ZZ.

```
function [t,y] = solveivp(t0, y0, WW, n)
  t = zeros(n+1,1); y = zeros(n+1,1);
  t(1) = t0; y(1) = y0; h = (WW-t0)/n;
  for j = 1:n
    t(j+1) = t(j)+h;
    y(j+1) = XX+h*(YY)^ZZ;
  end
```

The input values are $t0 = 0, y0 = 1, n = 100$ and the missing input WW. (2 p)

4. Let $\Omega = (0, 2)$ and $f \in L^2(\Omega)$. Consider the BVP

$$\begin{aligned} -u''(x) + u(x) &= f(x) \quad \text{for } x \in \Omega \\ u(0) = u(2) &= 0. \end{aligned}$$

(a) Provide the variational form of the above problem using a bilinear form denoted by $a(\cdot, \cdot)$. Do not forget to define the appropriate test and trial spaces. (2 p)

(b) Provide the FE problem for a cG(1) Galerkin approximation of the above variational formulation. Do not forget to define the appropriate test and trial spaces. (1 p)

(c) Provide the linear system of equations obtained from the FE problem in the case of two elements of length $h = 1$. (5 p)

(d) Denote by $u_h \in V_h^0$ the above cG(1) Galerkin approximation of the solution u to the variational formulation. Show that $u - u_h$ is orthogonal to V_h^0 with respect to the inner product coming from the bilinear form $a(\cdot, \cdot)$. We recall that this inner product is given by $(u, v)_a = a(u, v)$. Observe that this is Galerkin orthogonality in the present context. (2 p)

(e) Using the result from the previous step, prove that u_h is the best approximation of u in V_h^0 in the H^1 norm:

$$\|u - u_h\|_{H^1} \leq \|u - v_h\|_{H^1} \quad \text{for all } v_h \in V_h^0.$$

We recall that $H^1 = H^1(\Omega) = \{v \colon \Omega \to \mathbb{R} : \ v, v' \in L^2(\Omega)\}$ with the norm $\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|v'\|_{L^2}^2$. (3 p)

Hint: If you didn't do the previous step, you may use that $a(u - u_h, v_h) = 0$ for any $v_h \in V_h^0$ and then try to relate the H^1 norm to the inner product $(u, v)_a$.

5. Let $a > 0$ and $f\colon [0,a] \to \mathbb{R}$ continuous. Suppose that for all $x \in \mathbb{R}$ one has $\int_0^a f(t)\cos(xt)\,\mathrm{d}t = 0$. The goal of the exercise is to show that $f = 0$ (is the zero function).
   (a) For $t \in [-\pi,\pi]$ define $g(t) = f(a|t|/\pi)$ and extended 2π-periodically. Show that g is an even and continuous function. (2 p)
   (b) Compute the Fourier coefficients of g and show that they are all zero. (4 p)
   (c) From the above, deduce that $g = 0$ and hence $f = 0$. (1 p)
6. Is the function $f(t) = e^{5t}$, for $t \geq 0$, of exponential order α and piecewise continuous? How can you guarantee the existence of the Laplace transform of f? (3 p)

   Hint: You don't need to give too much details to answer this question.
7. Let $c_1, c_2 \in \mathbb{R}$ and two functions f_1, f_2 whose Laplace transforms exist for $s > \alpha$ for some α. Show that

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = c_1\mathcal{L}\{f_1(t)\}(s) + c_2\mathcal{L}\{f_2(t)\}(s). \qquad (2\text{ p})$$

8. Compute the Laplace transform of

$$f(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t). \qquad (3\text{ p})$$

9. Use the Laplace transform to solve the IVP

$$y''(t) + 3y'(t) + 2y(t) = 0, \quad y(0) = 1 \quad y'(0) = 0. \qquad (4\text{ p})$$

10. Explain how to use the superposition principle to solve the nonhomogeneous wave equation ($0 < x < 1$, $0 < t < T$)

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 77 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x, u_t(x,0) = x^2. \end{cases}$$

    Observe that there is no need to solve the subproblem(s). (2 p)
11. Consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

    on the interval $[0, 3\pi]$.
    (a) Using the separation of variables approach, obtain all solutions to the above heat equation with boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(3\pi,t) = 0$. (3 p)
    Hint: First, find two differential equations.

(b) Find the unique solution to the heat equation that satisfy the above boundary conditions and the initial condition

$$u(x,0) = 12\cos(5x) + 3\cos(8x). \qquad (2\text{ p})$$

Hint: If you cannot start, consider

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n}{3}x\right) e^{-(\frac{n}{3})^2 t}$$

and find the correct coefficients a_n.

Table of Laplace Transforms and trigonometry

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin(bt)-bt\cos(bt))$	$\frac{1}{(s^2+b^2)^2}$

$$
\begin{aligned}
2\sin(a)\sin(b) &= \cos(a-b)-\cos(a+b)\\
2\sin(a)\cos(b) &= \sin(a-b)+\sin(a+b)\\
2\cos(a)\cos(b) &= \cos(a-b)+\cos(a+b)
\end{aligned}
$$