(A)

TMA683: Exam 11.04.22

(1) a) Yes (indep. and span$(1, x, x^2) = \mathbb{P}^{(2)}(\mathbb{R})$) 1

b) The exact sol. to VF 1

c) $\int_0^1 x^2 dx \approx (1+0)\left(\frac{1+0}{2}\right)^2 = \frac{1}{4}$ 1

d) Yes 1

e) To possibly save cpu time and get accurate num. sol. 1

(2) a) $\lambda_i(x) = \prod_{j=0}^{n} \frac{x-x_j}{x_i-x_j}$ for $i=0,1,\dots,n$ 1

b) $\pi_n(x) = \sum_{j=0}^{n} f(x_j)\lambda_j(x)$ 1 of degree $\leq n$ 1

c) $\pi_n(x_j) = \sum_{j=0}^{n} f(x_j)\underbrace{\lambda_j(x_i)}_{\delta_{ij}} = f(x_i)$ ok 1

(3) WW=T=2 0.5, XX=y(j) 0.5, YY=y(j) 0.5, ZZ=2 0.5

(4) a) Multiply w/ test fct, ~~+~~ integration by parts give:

(VF) Find $u \in H_0^1(\Omega)$ s.t. $\underbrace{\int_0^2 u'(x)v'(x)dx + \int_0^2 u(x)v(x)dx}_{=a(u,v)} = \int_0^2 f(x)v(x)dx \quad \forall v \in H_0^1(\Omega)$ 2

trial=test=H_0^1 because of the BC

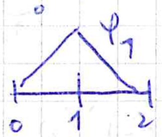
b) $V_h^0 = \text{span}(\varphi_j, j=1,2,\dots,n)$, φ_j hat fct on $[0,2]$:

(FE) Find $u_h \in V_h^0$ s.t. $a(u_h, v_h) = \int_0^2 f(x)v_h(x)dx \quad \forall v_h \in V_h^0$ 1

c) Take $u_h(x) = \sum_{j=1}^{n} \xi_j \varphi_j(x)$ 1, $v_h(x) = \varphi_i(x)$ 1 and get

$\sum_{j=1}^{n} a(\varphi_j, \varphi_i)$ 1 $\xi_j = \int_0^2 f(x)\varphi_i(x)dx$ 1 for $i=1,\dots,n$

$h=1 \to n=1 \to$



$\to A\xi = b$ with $b=(f,\varphi_1)$ $A=a(\varphi_1,\varphi_1)$ 1

d) To show $\underbrace{(u-u_h, v_h)_a}_{=a(u-u_h,v_h)} = 0 \quad \forall v_h \in V_h^0$

Recall $a(u,v) = \int fv\,dx \quad \forall v \in H_0^1$ 1

$a(u_h, v_h) = \int f v_h dx \quad \forall v_h \in V_h^0 \subset H_0^1$

Hence, taking $v = v_h$: $a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h^0$ by linearity

e) $\|u-u_h\|_{H^1}^2 \overset{\text{def } a(\cdot,\cdot)}{=} a(u-u_h, u-u_h) \underset{v_h \in V_h^0}{=} a(u-u_h, u-v_h+v_h-u_h) = a(u-u_h, u-v_h) + \underbrace{a(u-u_h, \underbrace{v_h-u_h}_{\in V_h^0})}_{=0 \text{ by } u_0}$ ①

$\overset{CS}{\leq} \|u-u_h\|_{H^1} \cdot \|u-v_h\|_{H^1} \Rightarrow \|u-u_h\|_{H^1} \leq \|u-v_h\|_{H^1} \quad \forall v_h \in V_h^0$ ①

5) a) g even $\Leftrightarrow g(-t) = g(t) \Leftrightarrow g(-t) = f(a|-t|/\pi) = f(a|t|/\pi) = g(t)$ OK ①

g continuous since f continuous ①

b) $a_0 = \frac{2}{\pi}\int_0^\pi g(t)\,dt$

$a_n = \frac{2}{\pi}\int_0^\pi g(t)\cos(nt)\,dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt$ ①

g even $\to b_n = 0 \ \forall n$ ①

$a_0 = \frac{2}{\pi}\int_0^\pi f(at/\pi)\,dt \underset{u = \frac{at}{\pi},\ du = \frac{a}{\pi}dt}{=} \frac{2}{\pi}\int_0^a f(u)\frac{a}{\pi}\,du = 0$ by assumption on f ①

same

$a_n = \frac{2}{\pi}\int_0^\pi f(at/\pi)\cos(nt)\,dt = \dots \quad \dots = 0$ by ass. on f ①

c) All Fourier coeff $= 0 \Rightarrow g \equiv 0 \underset{\text{Def } f/g}{\Rightarrow} f \equiv 0$ ①

6) $|f(t)| \leq 1 \cdot e^{\alpha t}$ with $\alpha = 5$ $\Rightarrow$ of exponential order ①

f continuous hence pwc ①

$\hookrightarrow$ by Th from lecture LT of f $\exists$. ①

7) LT = integral which are linear mappings ②

8) $\mathcal{L}\{f(t)\}(s) \underset{\text{linearity}}{=} \mathcal{L}\{e^{3t}\}(s) + \mathcal{L}\{\cos(6t)\}(s) - \mathcal{L}\{e^{3t}\cos(6t)\}(s) \underset{\text{Table}}{=} \frac{1}{s-3} + \frac{s}{s^2+6^2} - \frac{s-3}{(s-3)^2+6^2}$ ① ① ①

9) Take LT, $Y(s) = \mathcal{L}\{y(t)\}(s)$, and get $(s^2+3s+2)Y(s) = s+3$ ①

Get explicit form: $Y(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$ ① ①

Take ILT: $y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \mathcal{L}^{-1}\{\frac{2}{s+1}\}(t) - \mathcal{L}^{-1}\{\frac{1}{s+2}\}(t) = 2e^{-t} - e^{-2t}$ ①

(10) Sol to this wave eq. can be obtained as $u = v + w$, where v sol. homogeneous wave eq. (1) and w sol. to BVP (1). That is

$$\begin{cases} v_{tt} - v_{xx} = 0 \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = \underset{x}{f(x)} - w(x),\ v_t(x,0) = \underset{x^2}{g(x)} \end{cases} \qquad \begin{cases} -w''(x) = 77 \\ w(0,t) = w(1,t) = 0 \end{cases}$$

(11) a) Set $u(x,t) = X(x) \cdot T(t)$ with $\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{T(t)} = -\lambda^2$, hence

$\dot{T}(t) + \lambda^2 T(t) = 0$ (0,5) $\rightarrow T(t) = C \exp(-\lambda^2 t)$ (0,5)

$X''(x) + \lambda^2 X(x) = 0$ (0,5) $\rightarrow X_n(x) = D_n \cos\left(\frac{n}{3} x\right)$ for $n = 0, 1, 2, \ldots$ (0,5)

B.C. $\frac{\partial u}{\partial x}(3\pi, t) = \frac{\partial u}{\partial x}(9, t) = 0$

$$\longrightarrow u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n}{3} x\right) e^{-\left(\frac{n}{3}\right)^2 t} \quad (1)$$

superposition

b) $12\cos(5x) + 3\cos(8x) \underset{\text{IC}}{=} u(x,0) \underset{\text{a)/Hint}}{=} \sum_{n=0}^{\infty} a_n \cos\left(\frac{n}{3} x\right)$ (1) $\underset{\text{Fourier series}}{\Longrightarrow} \begin{cases} a_{15} = 12 \\ a_{24} = 3 \\ a_n = 0 \text{ for } n \neq 15, 24 \end{cases}$ (1)