TMA683 Tillämpad matematik
Övningsuppgifter (boken FEM)

28 oktober 2022

This document contains the exercises from the compendium from M. Asadzadeh (23.08.2018). Particularly relevant exercises are marked with $(*)$.

Propositions or hints for solutions are given at the end of the file (thanks to Sebastian Persson).

Thank you for reporting typos or errors via email.

1. Chapter 4: Polynomial approximation in $1d$

4.1 Prove that $V_0^{(q)} = \{v \in \mathcal{P}^{(q)}(0,1), v(0) = 0\}$ is a subspace of $\mathcal{P}^{(q)}(0,1)$.

4.3 Consider the ODE

$$\dot{u}(t) = u(t), \quad 0 < t < 1, \quad u(0) = 1.$$

Compute its Galerkin approximation in $\mathcal{P}^{(q)}(0,1)$ for $q = 1, 2, 3, 4$.

4.4 $(*)$ Compute the stiffness matrix and load vector in a finite element approximation of the BVP

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

with $f(x) = x$ and $h = 1/4$.

4.5 We want to find a solution approximation $U(x)$ to

$$-u''(x) = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

using the ansatz $U(x) = A\sin(\pi x) + B\sin(2\pi x)$.

(a) Calculate the exact solution $u(x)$.

(b) Write down the residual $R(x) = -U''(x) - 1$.

(c) Use the orthogonality condition

$$\int_0^1 R(x)\sin(n\pi x)\,\mathrm{d}x = 0, n = 1, 2$$

to determine the constants A and B.

(d) Plot the error $e(x) = |u(x) - U(x)|$.

4.6 Consider the BVP

$$-u''(x) + u(x) = x, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

(a) Verify that the exact solution to the above problem reads

$$u(x) = x - \frac{\sinh(x)}{\sinh(1)}.$$

(b) Let $U(x)$ be a solution approximation defined by

$$U(x) = A\sin(\pi x) + B\sin(2\pi x) + C\sin(3\pi x),$$

where A, B, C are unknown constants. Compute the residual

$$R(x) = -U''(x) + U(x) - x.$$

(c) Use the orthogonality conditions

$$\int_0^1 R(x)\sin(n\pi x)\,\mathrm{d}x = 0, n = 1, 2, 3$$

to determine the constants A, B, C.

4.7 Let $U(x) = \zeta_0\phi_0(x) + \zeta_1\phi_1(x)$ be a solution approximation to

$$-u''(x) = x - 1, \quad 0 < x < \pi, \quad u'(0) = u(\pi) = 0,$$

where ζ_0 and ζ_1 are unknown coefficients and $\phi_0(x) = \cos(\frac{x}{2})$, $\phi_1(x) = \cos(\frac{3x}{2})$.

(a) Find the analytical solution $u(x)$.

(b) Define the residual $R(x)$.

(c) Compute the constants ζ_0 and ζ_1 using the orthogonality conditions

$$\int_0^\pi R(x)\phi_i(x)\,\mathrm{d}x = 0, i = 0, 1.$$

I.e. by projecting $R(x)$ onto the vector space spanned by $\phi_0(x)$ and $\phi_1(x)$.

4.8 Use the projection technique of the previous exercise to solve

$$-u''(x) = 0, \quad 0 < x < \pi, \quad u(0) = 0, u(\pi) = 2,$$

with $U(x) = A\sin(x) + B\sin(2x) + C\sin(3x) + \frac{2}{\pi^2}x^2$ and using the test functions $\{\sin(x), \sin(2x), \sin(3x)\}$.

2. Chapter 5: Interpolation, Numerical integration in $1d$

5.1 Consider two real numbers $a < b$. By defintion of Lagranges polynomials, one has

$$\lambda_a(x) = \frac{b-x}{b-a} \quad \text{and} \quad \lambda_b(x) = \frac{x-a}{b-a}.$$

Show that

$$\lambda_a(x) + \lambda_b(x) = 1 \quad \text{and} \quad a\lambda_a(x) + b\lambda_b(x) = x.$$

Give a geometric interpretation by plotting $\lambda_a(x), \lambda_b(x), \lambda_a(x)+\lambda_b(x)$ and $a\lambda_a(x), b\lambda_b(x), a\lambda_a(x)+b\lambda_b(x)$.

5.2 $(*)$ Consider the following functions defined for $x \in [0,1]$:

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sin(\pi x).$$

Find their linear interpolants, denoted by $\Pi f \in \mathcal{P}(0,1)$, resp. $\Pi g \in \mathcal{P}(0,1)$. In the same figure, plot f and Πf, as well as g and Πg.

5.3 Determine the linear interpolant of the function, defined for $x \in [-\pi, \pi]$,

$$f(x) = \frac{1}{\pi^2}(x-\pi)^2 - \cos^2(x - \frac{\pi}{2}),$$

where the interval $[-\pi, \pi]$ is divided into 4 equal subintervals.

5.15 Prove that

$$\int_{x_0}^{x_1} f'(\frac{x_0+x_1}{2})(x - \frac{x_0+x_1}{2}) \,\mathrm{d}x = 0.$$

5.16 $(*)$ Prove that

$$\begin{aligned}\left|\int_{x_0}^{x_1} f(x)\,\mathrm{d}x - f(\frac{x_0+x_1}{2})(x_1-x_0)\right| &\leq \frac{1}{2}\max_{[x_0,x_1]}|f''(x)| \int_{x_0}^{x_1}(x - \frac{x_0+x_1}{2})^2\,\mathrm{d}x \\ &\leq \frac{1}{24}(x_1-x_0)^3 \max_{[x_0,x_1]}|f''(x)|.\end{aligned}$$

Hint: Use a Taylor expansion of f about $x = \frac{x_0+x_1}{2}$.

3. Chapter 7: Two-point boundary value problems

7.1 Consider the two-point BVP

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

Let $V = \{v\colon \|v\| + \|v'\| < \infty, v(0) = v(1) = 0\}$ where $\|\cdot\|$ denotes the L_2-norm.

(a) Use V to derive a variational formulation for the above BVP.

(b) Discuss why V is valid as a vector space of test functions.

(c) Classify which of the following functions are admissible test functions:

$$\sin(\pi x), \quad x^2, \quad x\ln(x), \quad \mathrm{e}^x - 1, \quad x(1-x).$$

7.3 Consider the two-point BVP

$$-u''(x) = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

Let $\mathcal{T}_h : x_j = \frac{j}{4}, j = 0, 1, 2, 3, 4$ denote a partition of the interval $0 < x < 1$ into four subintervals of equal length $h = 1/4$. Let V_h be the corresponding space of continuous piecewise liner functions vanishing at $x = 0$ and $x = 1$.

(a) Compute a finite element approximation $U \in V_h$ to the above BVP.

(b) Prove that $U \in V_h$ is unique.

7.5 $(*)$ Consider the two-point BVP, for $x \in I = (0, 1)$:

$$-(a(x)u'(x))' = f(x)$$
$$u(0) = 0, \quad a(1)u'(1) = g_1,$$

where a is a positive function and g_1 a constant.

(a) Derive the variational formulation of the above problem.

(b) Discuss how the boundary conditions are implemented.

7.6 Consider the two-point BVP, for $x \in I = (0, 1)$,

$$-u''(x) = 0$$
$$u(0) = 0, u'(1) = 7.$$

Divide the interval I into two subintervals of length $h = \frac{1}{2}$. Let V_h be the corresponding space of continuous piecewise linear functions vanishing at $x = 0$.

(a) Formulate a finite element method for the above problem.

(b) Calculate by hand the finite element approximation $U \in V_h$ to the above BVP.

(c) Study how the boundary condition at $x = 1$ is approximated.

7.7 (∗) Consider the two-point BVP

$$-u''(x) = 0, \quad 0 < x < 1, \quad u'(0) = 5, u(1) = 0.$$

Let $\mathcal{T}_h : x_j = \frac{j}{N}, j = 0, 1, \dots, N, h = 1/N$ denote a uniform partition of the interval $0 < x < 1$ into N subintervals. Let V_h be the corresponding space of continuous piecewise linear functions.

(a) Use V_h, with $N = 3$, and formulate a finite element method for the above problem.

(b) Compute the finite element approximation $U \in V_h$ assuming $N = 3$.

7.8 Consider the problem of finding a solution approximation to

$$-u''(x) = 1, \quad 0 < x < 1, \quad u'(0) = u'(1) = 0.$$

Let $\mathcal{T}_h$ be a partition of the interval $0 < x < 1$ into two subintervals of equal length $h = \frac{1}{2}$. Let V_h be the corresponding space of continuous piecewise linear functions.

(a) Can you find an exact solution to the above problem by integrating twice?

(b) Compute a finite element approximation $U \in V_h$ to u if possible.

7.11 Consider the finite element method applied to

$$-u''(x) = 0, \quad 0 < x < 1, \quad u(0) = \alpha, u'(1) = \beta,$$

where α and β are given constants. Assume that the interval $[0, 1]$ is divided into three subintervals of equal length $h = 1/3$ and that $\{\varphi_j\}_{j=0}^3$ is a nodal basis of V_h, the corresponding space of continuous piecewise linear functions.

(a) Verify that the ansatz

$$U(x) = \alpha\varphi_0(x) + \zeta_1\varphi_1(x) + \zeta_2\varphi_2(x) + \zeta_3\varphi_3(x),$$

yields the following system of equations

$$\frac{1}{h}\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}. \tag{1}$$

(b) If $\alpha = 2$ and $\beta = 3$ show that (1) can be reduced to

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 2h^{-1} \\ 0 \\ 3 \end{pmatrix}.$$

(c) Solve the above system of equation to find $U(x)$.

7.13 Consider the following boundary value problem

$$-au''(x) + bu(x) = 0, \quad 0 \le x \le 1, \quad u(0) = u'(1) = 0,$$

where $a, b > 0$ are constants. Let $\mathcal{T}_h : 0 = x_0 < x_1 < \ldots < x_N = 1$, be a non-uniform partition of the interval $0 \le x \le 1$ into N intervals of length $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. Let V_h be the corresponding space of continuous piecewise linear functions. Compute the stiffness and mass matrices.

7.14 Show that the FEM with mesh size h for the problem

$$\begin{cases} -u''(x) = 1 & 0 < x < 1 \\ u(0) = 7, u'(1) = 0, \end{cases}$$

with $U(x) = 7\varphi_0(x) + U_1\varphi_1(x) + \ldots + U_m\varphi_m(x)$ leads to the linear system of equations $\tilde{A}\tilde{U} = \tilde{b}$, where $\tilde{A} \in \mathbb{R}^{m\times(m+1)}$, $\tilde{U} \in \mathbb{R}^{(m+1)\times 1}$, $\tilde{b} \in \mathbb{R}^{m\times 1}$ are given by

$$\tilde{A} = \frac{1}{h}\begin{pmatrix} -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tilde{U} = \begin{pmatrix} 7 \\ U_1 \\ \vdots \\ U_m \end{pmatrix}, \tilde{b} = \begin{pmatrix} h \\ \vdots \\ h \\ h/2 \end{pmatrix}.$$

The above reduces to $AU = b$, with

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & -1 & 2 \end{pmatrix}, U = \begin{pmatrix} U_1 \\ \vdots \\ U_m \end{pmatrix}, b = \begin{pmatrix} h + \frac{7}{h} \\ \vdots \\ h \\ h/2 \end{pmatrix}.$$

4. Chapter 8: Scalar initial value problems

8.5a) Compute the solution of

$$\dot{u}(t) + a(t)u(t) = t^2, \quad 0 < t < T, \quad u(0) = 1,$$

where $a(t) = 4$.

5. Chapter 9: Initial boundary value problems in $1d$

9.7 Consider the inhomogeneous problem

$$\begin{cases} u_t(x,t) - \varepsilon u_{xx}(x,t) = f(x,t), & 0 < x < 1, t > 0 \\ u(0,t) = u_x(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

Show that for the corresponding stationary problem, $u_t = 0$, one has

$$\|u_x\| \leq \frac{1}{\varepsilon} \|f\|.$$

9.13 Consider the wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \\ u_t(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases}$$

Plot the graph of $u(x,2)$ in the following cases:

(a) $v_0 = 0$ and

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0. \end{cases}$$

(b) $u_0 = 0$ and

$$v_0(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ 0, & |x| > 1. \end{cases}$$

6. Chapter 4: Propositions for solutions

4.1 Use the definitions of $\mathcal{P}^{(q)}(0,1)$ and of a subspace.

4.3 Every element $v \in \mathcal{P}^{(q)}(0,1)$ can be written as

$$v(t) = \sum_{j=0}^{q} \chi_j t^j.$$

Use this in a VF of the problem.

4.4 See the lecture.

4.5 (a) The exact solution reads $u(x) = \frac{x}{2}(1-x)$.

(b) The residual reads $R(x) = \pi^2 \left(A\sin(\pi x) + 4B\sin(2\pi x)\right) - 1$.

(c) $A = \frac{4}{\pi^3}$ and $B = 0$.

4.6 (a) ok

(b)

$$R(x) = (\pi^2+1)A\sin(\pi x) + (4\pi^2+1)B\sin(2\pi x) + (9\pi^2+1)C\sin(3\pi x) - x.$$

(c)

$$A = \frac{2}{\pi(\pi^2+1)}, B = -\frac{1}{\pi(4\pi^2+1)}, C = \frac{2}{3\pi(9\pi^2+1)}$$

4.7 (a)

$$u(x) = \frac{1}{6}(\pi^3 - x^3) + \frac{1}{2}(x^2 - \pi^2)$$

(b)

$$R(x) = -U''(x) - x + 1 = \frac{1}{4}\zeta_0 \cos(\frac{x}{2}) + \frac{9}{4}\zeta_1 \cos(\frac{3x}{2}) - x + 1$$

(c)

$$\zeta_0 = 8(2\pi - 6)/\pi, \zeta_1 = \frac{8}{9}(\frac{2}{9} - \frac{2}{3}\pi)/\pi$$

4.8

$$U(x) = (16\sin(x) + \frac{16}{27}\sin(3x))/\pi^3 + \frac{2}{\pi^2}x^2$$

7. Chapter 5: Propositions for solutions

5.1 Insert the definition of

$$\lambda_a(x) = \frac{b-x}{b-a} \quad \text{and} \quad \lambda_b(x) = \frac{x-a}{b-a}.$$

into

$$\lambda_a(x) + \lambda_b(x) \quad \text{and} \quad a\lambda_a(x) + b\lambda_b(x)$$

to answer the exercise.

5.2 Use the definition of the linear interpolant, see lecture.

5.3

$$\Pi_1 f(x) = \begin{cases} 4 - 11(x+\pi)/(2\pi), & -\pi \leq x \leq -\frac{\pi}{2} \\ 5/4 - (x+\frac{\pi}{2})/(2\pi), & -\frac{\pi}{2} \leq x \leq 0 \\ 1 - 7x/(2\pi), & 0 \leq x \leq \frac{\pi}{2} \\ 3(x-\pi)/(2\pi), & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

5.15 Observe that the term $f'(\frac{x_0+x_1}{2})$ does not depend on x and use the formula $(a+b)(a-b) = a^2 - b^2$.

5.16 This is the local error of the midpoint rule. Use a Taylor expansion (with rest term) of f about $x = \frac{x_0+x_1}{2}$ to show the exercise.

8. Chapter 7: Propositions for solutions

7.1 (a) See lecture.
(b) See lecture.
(c) The following functions are admissible test functions:

$$\sin(\pi x), \quad x(1-x).$$

7.3 (a) See lecture.
(b) Assume that one has more than one solution to the FE and, using the FE formulation, find a contradiction.

7.5 (a) Similar to the lecture.
(b) Consider possible additional terms in the last vector.

7.6 (a) Similar to the lecture.
(b) Long computation . . . .
(c) tba

7.7 (a) Find $u_h \in V_h$ such that

$$\int_0^1 u_h(x) v_h(x) \, \mathrm{d}x = -5 v_h(0)$$

for all $v \in V_h^0$.
(b) The FE solution reads

$$u_h(x) = \alpha_0 \varphi_0(x) + \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x),$$

where φ_j are the hat functions and $\alpha_0 = -5, \alpha_1 \approx -3.333, \alpha_2 \approx -1.667$.

7.8 (a) Integrate the problem twice and do not forget the two integration constants.
(b) Observe that the resulting matrix from a FE discretisation is not invertible.

7.11 (a) Observe that one has non-homogeneous Dirichlet BC and hence need two spaces (trial, resp. test)

$$V = \{v\colon v, v' \in L^2(0,1), v(0) = \alpha\} \quad \text{and} \quad V^0 = \{v\colon v, v' \in L^2(0,1), v(0) = 0\}$$

for the VF (similarly for the FE formulation).
(b) ok
(c) One can use matlab to compute such solution.

7.13 Similar to the lecture.

7.14 Similar to the lecture.

9. Chapter 8: Propositions for solutions

8.5a)

$$u(t) = \mathrm{e}^{-4t} + \frac{1}{32}(8t^2 - 4t + 1)$$

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10. Chapter 9: Propositions for solutions

9.7 Recall the definition of the L^2-norm:

$$\|u\|^2 = (u, u) = \int_0^L u(x)u(x)\,\mathrm{d}x$$

and multiply the problem with an appropriate function and integrate (in space). Poincaré inequality could also be of some use.

9.13 One may use d'Alembet's formula (wiki)

$$u(x,t) = \frac{1}{2}(u_0(x-t) - u_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,\mathrm{d}y.$$