Chapter 1: Introduction and motivation (summary)

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Goal: Ordinary and partial differential equations are widely used to model various phenomena in sciences and engineering. This chapter sets basic definitions and notation.

- A differential equation (DE) is an equation that relates an unknown function (or more) and its derivative(s).
- An ordinary differential equation (ODE) is a DE, where the unknown function depends only on *one* variable (say $y(x)$ or $x(t)$ for instance).
- To determine a unique solution to an ODE, one needs to specify additional conditions. Two typical examples are:

  An initial value problem (IVP) consists of an ODE with an initial value or initial condition. The Malthusian growth model for bacteria reads (t_0, T, P_0 and λ are given, $t \in [t_0, T]$ and $P(t)$ is unknown)

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} P(t) = \lambda P(t) \\ P(t_0) = P_0. \end{cases}$$

  Here, P_0 is the size of the initial population of bacteria and $P(t)$ would describe the size of the population at time t.

  A boundary value problem (BVP) consists of an ODE with boundary conditions. For instance ($u(x)$ is unknown)

$$\begin{cases} -u''(x) = \cos(x) \quad \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} \quad u(1) = 5. \end{cases}$$

  Here, one specifies the values of the solution $u(x)$ at the boundaries 0 and 1.

- A partial differential equation (PDE) is a DE, where the unknown function depends on *2 or more* (independent) variables (say $u(x,y)$ or $u(t,x,y,z)$ for instance).
- Typical examples of PDEs are

  Laplace's equation

$$\Delta u = 0,$$

  with the Laplace operator Δ defined by $\Delta u(x) = \sum_{k=1}^{n} u_{x_k,x_k}(x)$ for $x = (x_1, x_2, \dots, x_n)$ and $u\colon \mathbb{R}^n \to \mathbb{R}$. In $2d$, Laplace's equation reads $u_{x_1,x_1}(x_1,x_2) + u_{x_2,x_2}(x_1,x_2) = 0$.

  The heat equation (observe that the derivative(s) in the Laplacian are related to the variable(s) in space)

$$u_t - \Delta u = f.$$

  In $1d$, the above reads $u_t(x,t) - u_{xx}(x,t) = f(x)$, where $u(x,t)$ could describe the temperature at time t and position x of a thin wire.

  The wave equation

$$u_{tt} - \Delta u = g.$$

  In $1d$, the above reads $u_{tt}(x,t) - u_{xx}(x,t) = g(x)$, where $u(x,t)$ could describe the motion of a guitar string at time t and position x (on the string).

- To specify a unique solution to a PDE, one also needs additional conditions. For the example of the $1d$ heat equation on $[0,1]$, one gets

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x) \quad \text{for} \quad x \in (0,1), t \in (0,T] \\ u(0,t) = 0, u(1,t) = \sin(t) \quad \text{for} \quad t \in (0,T] \\ u(x,0) = 3x \quad \text{for} \quad x \in (0,1). \end{cases}$$

  The last condition is an initial value/condition (it specifies the initial temperature profile). The other conditions are boundary conditions (they specify the value of the temperature at the ends of the wire). Observe that one has one initial value (since the DE has one derivative in time) and two boundary conditions (since the DE has two derivatives in space).

Further resources (clickable links):

- wikipedia.org/DE
- sv.wikipedia.org/ODE
- sv.wikipedia.org/PDE
- tutorial.math.lamar.edu/DE1
- tutorial.math.lamar.edu/DE2
- khanacademy.org/DE
- analyzemath.com/DE