Chapter 3: Interpolation and numerical integration (summary)

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Goals:
Interpolation: We want to pass a (simple) function through a given set of data points.

Numerical integration: We want to find numerical approximations of integrals $\int_a^b f(x)\,\mathrm{d}x$.

- Let $q \in \mathbb{N}$ and $a < b$. Consider a continuous function $f\colon [a,b] \to \mathbb{R}$ and $(q+1)$ distinct interpolation points $\big(x_j, f(x_j)\big)_{j=0}^q$ with $a = x_0 < x_1 < \ldots < x_q = b$. A polynomial $\pi_q f \in \mathscr{P}^{(q)}(a,b)$ is an interpolant for f if
$$\pi_q f(x_j) = f(x_j) \quad \text{for} \quad j = 0,1,2,\ldots,q.$$
Example: Remembering that $\mathscr{P}^{(q)}(a,b) = \operatorname{span}(1,x,x^2,\ldots,x^q)$, one gets the polynomial interpolant
$$\pi_q f(x) = \sum_{j=0}^q a_j x^j.$$
The coefficients a_j are then found using the conditions $\pi_q f(x_j) = f(x_j)$ for $j = 0,1,\ldots,q$.

- Consider an interval $[a,b]$ and a grid of $(q+1)$ distinct points $x_0 = a < x_1 < \ldots < x_q = b$. One defines Lagrange polynomials by
$$\lambda_i(x) = \prod_{j=0, j\neq i}^q \frac{x - x_j}{x_i - x_j}$$
for $i = 0,1,\ldots,q$. One then has (no proof)
$$\mathscr{P}^{(q)}(a,b) = \operatorname{span}\big(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x)\big).$$
The above permits to find the interpolant of f using another basis.
Example: Taking $\mathscr{P}^{(q)}(a,b) = \operatorname{span}(\lambda_0(x), \lambda_1(x), \ldots, \lambda_q(x))$, one gets the Lagrange interpolant
$$\pi_q f(x) = \sum_{j=0}^q f(x_j)\lambda_j(x),$$
where λ_j are the Lagrange polynomials defined above. Obs: This gives the same interpolant polynomial has above.

- Under some assumptions on the function f, the error of the linear interpolant $\pi_1 f$ is given by
$$\|\pi_1 f - f\|_{L^p(a,b)} \leq C h^2 \|f''\|_{L^p(a,b)},$$
for $p = 1,2$ or ∞. Other error bounds have been seen in the lecture.

- Denote a partition of the interval $[0,1]$ into $m+1$ subintervals by $\tau_h : 0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = 1$, where $h_j = x_j - x_{j-1}$ for $j = 1,2,\ldots,m+1$. We define the hat function $\{\varphi_j\}_{j=0}^{m+1}$ by
$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h_j} & \text{for } x_{j-1} \leq x \leq x_j \\ \frac{x - x_{j+1}}{-h_{j+1}} & \text{for } x_j \leq x \leq x_{j+1} \\ 0 & \text{else} \end{cases}$$

for $j = 1, \ldots, m$. The functions $\varphi_0(x)$ and $\varphi_{m+1}(x)$ are defined as half hat functions.

With the above, one then defines the space of continuous piecewise linear functions on $[0,1]$ by

$$V_h = V_h(0,1) = \left\{v\colon [0,1] \to \mathbb{R} \quad : \ v \text{ cont. piecewise linear on } \tau_h\right\} = \operatorname{span}\left(\varphi_0, \varphi_1, \ldots, \varphi_{m+1}\right).$$

As usual, one has $v(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x)$, where $\zeta_j = v(x_j)$, for any $v \in V_h$.

- Consider a uniform partition of an interval $[a,b]$, denoted $\tau_h : x_0 = a < x_1 < \ldots < b = x_{m+1}$, and the space of continuous piecewise linear functions on τ_h, $V_h = \operatorname{span}(\varphi_0, \ldots, \varphi_{m+1})$ with hat functions φ_j. The continuous piecewise linear interpolant of f is defined by

$$\pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\varphi_j(x) \quad \text{for} \quad x \in [a,b].$$

  If $f \in \mathscr{C}^2(a,b)$ one has, for instance, the following bound for the interpolation error for the continuous piecewise linear interpolant

$$\left\|\pi_h f - f\right\|_{L^p(a,b)} \leq Ch^2 \left\|f''\right\|_{L^p(a,b)},$$

  for $p = 1, 2$ or ∞.

- Let us give 3 classical quadrature rules to numerically approximate the integral $\int_a^b f(x)\,\mathrm{d}x$:

  The midpoint rule reads

$$\int_a^b f(x)\,\mathrm{d}x \approx (b-a) f\left(\frac{a+b}{2}\right).$$

  The trapezoidal rule reads

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{2}\left(f(a) + f(b)\right).$$

  The Simpson rule reads

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

  In practice, one first considers a (uniform) partition of the interval $[a,b]$, $a = x_0 < x_1 < \ldots < x_N = b$, where N is a given (large) integer. One then apply a quadrature rule (denoted by $QF(x_j, x_{j+1}, f)$ below) on each small subintervals:

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} f(x)\,\mathrm{d}x \approx \sum_{j=0}^{N-1} QF(x_j, x_{j+1}, f).$$

Further resources:

- www.dcode.fr
- www.maths.lth.se1, www.maths.lth.se2, www.maths.lth.se3.
- www.phys.libretexts.org
- www.khanacademy.org
- tutorial.math.lamar.edu