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Chapter 5: FEM for two-point BVP (summary)

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Goal: We use the theoretical and practical tools from the previous chapters to present and analyse FEM for several BVP.

- For a positive integer q and $f \in L^2(a,b)$, one defines its L^2-projection as the polynomial $Pf \in \mathscr{P}^{(q)}(a,b)$ verifying

$$\int_a^b f(x)p(x)\,\mathrm{d}x = \int_a^b (Pf)(x)p(x)\,\mathrm{d}x \quad \text{for all} \quad p \in \mathscr{P}^{(q)}(a,b)$$

or shortly

$$(f,p)_{L^2(a,b)} = (Pf,p)_{L^2(a,b)} \quad \text{for all} \quad p \in \mathscr{P}^{(q)}(a,b)$$

or (since monomials x^j are basis of $\mathscr{P}^{(q)}(a,b)$)

$$(f,x^j)_{L^2(a,b)} = (Pf,x^j)_{L^2(a,b)} \quad \text{for} \quad j=0,1,\dots,q.$$

Theoretical results: The L^2-projection Pf is unique and the best approximation of f in $\mathscr{P}^{(q)}(a,b)$ in the L^2-norm.

- In a nutshell, a Galerkin finite element method (FEM) for the BVP with homogeneous Dirichlet BC

$$\begin{cases} -u''(x) = f(x) \quad \text{for} \quad x \in (0,1) \\ u(0)=0, u(1)=0 \end{cases}$$

consists of the following

1. Multiply the DE by a test function $v \in V^0 = \{v\colon [0,1] \to \mathbb{R}\colon\ v, v' \in L^2(0,1) \text{ and } v(0)=v(1)=0\}$.
2. Integrate the above over the domain $[0,1]$ and get the variational formulation of the problem (VF)

$$\text{Find} \quad u \in V^0 \quad \text{such that} \quad \int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \text{for all} \quad v \in V^0$$

or shortly

$$\text{Find} \quad u \in V^0 \quad \text{such that} \quad (u',v')_{L^2(0,1)} = (f,v)_{L^2(0,1)} \quad \forall v \in V^0.$$

3. Specify the finite dimensional space $V_h^0 \subset V^0$ defined as $V_h^0 = \operatorname{span}(\varphi_1,\dots,\varphi_m)$, for the hat functions φ_j defined on a uniform partition of $[0,1]$ with mesh $h = \frac{1}{m+1}$. Consider the FE problem

$$\text{Find} \quad u_h \in V_h^0 \quad \text{such that} \quad (u_h',v_h')_{L^2(0,1)} = (f,v_h)_{L^2(0,1)} \quad \forall v_h \in V_h^0.$$

4. Insert the ansatz

$$u_h(x) = \sum_{j=1}^m \zeta_j \varphi_j(x)$$

into the FE problem and take $v_h = \varphi_i$, for $i=1,\dots,m$, to get a linear system of equation for the unknown $\zeta = (\zeta_1,\dots,\zeta_m)$:

$$S\zeta = b.$$

Here, S is termed the stiffness matrix (with entries $s_{ij} = (\varphi_i', \varphi_j')_{L^2(0,1)}$ for $i, j = 1, \ldots, m$) and b the load vector (with entries $b_i = (f, \varphi_i)_{L^2(0,1)}$ for $i = 1, \ldots, m$). Solving this linear system of equations gives us the vector ζ and then the FE approximation

$$u_h(x) = \sum_{j=1}^{m} \zeta_j \varphi_j(x)$$

to the exact solution u of the above BVP!

- In order to get a FE approximation to the BVP ($a > 0$ and f are (nice and) given)

$$\begin{cases} -(a(x)u'(x))' = f(x) & \text{for} \quad x \in (0,1) \\ u(0) = 0 \quad \text{and} \quad u(1) = 0 \end{cases}$$

  we proceed as usual:

  1. Define the test/trial space $H_0^1 = \{v\colon [0,1] \to \mathbb{R} :\ v, v' \in L^2(0,1), v(0) = v(1) = 0\}$, multiply the DE with a test function $v \in H_0^1$, integrate over the domain $[0,1]$ and get the VF

  $$\text{Find } u \in H_0^1 \quad \text{such that} \quad \int_0^1 a(x)u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1.$$

     Observe that the trial and test spaces are the same since the BVP has homogeneous Dirichlet BC.
  2. Define the finite dimensional space $V_h^0 = \{v\colon [0,1] \to \mathbb{R} :\ v \text{ is cont. pw. linear on } T_h, v(0) = v(1) = 0\}$, where as usual T_h is a uniform partition with mesh $h = \frac{1}{m+1}$.
     Observe that $V_h^0 = \operatorname{span}(\varphi_1, \ldots, \varphi_m) \subset H_0^1$ with the hat functions φ_j.
     The FE problem then reads

  $$\text{Find } u_h \in V_h^0 \quad \text{such that} \quad \int_0^1 a(x)u_h'(x)v_h'(x)\,\mathrm{d}x = \int_0^1 f(x)v_h(x)\,\mathrm{d}x \quad \forall v_h \in V_h^0.$$

     The above is also called $cG(1)$ FE (for linear continuous Galerkin FE).
  3. Choosing $v_h = \varphi_i$ for $i = 1, \ldots, m$, writing $u_h(x) = \sum_{j=1}^{m} \zeta_j \varphi_j(x) \in V_h^0$, and inserting everything into the FE problem gives the following linear system of equations

  $$S\zeta = b,$$

     where the $m \times m$ stiffness matrix S has entries $s_{ij} = \int_0^1 a(x)\varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$ and the $m \times 1$ load vector b has entries $b_i = \int_0^1 f(x)\varphi_i(x)\,\mathrm{d}x$. Formulas for these entries can be found in the book. Solving this system gives the vector ζ and in turns the FE approximation u_h.

- The above needs minor adaptations when dealing with other BC.
  Let us for example derive a FE approximation for the following BVP

$$\begin{cases} -u''(x) + 4u(x) = 0 & \text{for} \quad x \in (0,1) \\ u(0) = \alpha \quad \text{and} \quad u(1) = \beta, \end{cases}$$

  where $\alpha \neq 0$ and $\beta \neq 0$ are given real number. Such boundary conditions are called non-homogeneous Dirichlet boundary conditions.
  The derivation of a numerical approximation for solutions to the above problem is given by

1. Define the trial space $V=\{v\colon [0,1]\to\mathbb{R}:\ v,v'\in L^2(0,1), v(0)=\alpha, v(1)=\beta\}$ and the test space $V^0=\{v\colon [0,1]\to\mathbb{R}:\ v,v'\in L^2(0,1), v(0)=v(1)=0\}$. Multiply the DE with a test function $v\in V^0$, integrate over the domain $[0,1]$ and get the VF
$$\text{Find } u\in V \quad \text{such that} \quad \int_0^1 u'(x)v'(x)\,\mathrm{d}x+4\int_0^1 u(x)v(x)\,\mathrm{d}x=0 \quad \forall v\in V^0.$$
2. Next, define the finite dimensional spaces
$V_h=\{v\colon [0,1]\to\mathbb{R}:\ v \text{ is cont. pw. linear on } T_h \text{ and} v(0)=\alpha, v(1)=\beta\}$ and
$V_h^0=\{v\colon [0,1]\to\mathbb{R}:\ v \text{ is cont. pw. linear on } T_h, v(0)=v(1)=0\}$, where as before T_h is a uniform partition with mesh $h=\frac{1}{m+1}$. Observe that $V_h=\operatorname{span}(\varphi_0,\varphi_1,\ldots,\varphi_m,\varphi_{m+1})\subset V$ and $V_h^0=\operatorname{span}(\varphi_1,\ldots,\varphi_m)\subset V^0$ with the hat functions φ_j.
The FE problem then reads
$$\text{Find } u_h\in V_h \quad \text{such that} \quad \int_0^1 u_h'(x)v_h'(x)\,\mathrm{d}x+4\int_0^1 u_h(x)v_h(x)\,\mathrm{d}x \quad \forall v_h\in V_h^0.$$
3. Choosing $v_h=\varphi_i$, writing $u_h(x)=\sum_{j=0}^{m+1}\zeta_j\varphi_j(x)$ with $\zeta_0=\alpha$ and $\zeta_{m+1}=\beta$ (due to the non-homogeneous Dirichlet BC), and inserting everything into the FE problem gives the following linear system of equations
$$(S+4M)\zeta=b,$$
where the $m\times m$ stiffness matrix S has entries $s_{ij}=\int_0^1\varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, see above for details, the $m\times m$ mass matrix M has entries $m_{ij}=\int_0^1\varphi_i(x)\varphi_j(x)\,\mathrm{d}x$, and the $m\times 1$ vector b has entries $b_i=-\alpha(\varphi_0',\varphi_i')_{L^2}-\beta(\varphi_{m+1}',\varphi_i')_{L^2}-4\alpha(\varphi_0,\varphi_i)_{L^2}-4\beta(\varphi_{m+1},\varphi_i')_{L^2}$. The entries of the matrices S and M as well as of the vector b can be computed exactly.
Solving this system gives the vector ζ and in turns the FE approximation u_h.

- Let us finally consider the problem of finding a numerical approximation of solutions to the BVP
$$\begin{cases}-au''(x)+bu'(x)=r & \text{for} \quad x\in(0,1)\\ u(0)=0 \quad \text{and} \quad u'(1)=\beta,\end{cases}$$
where $\beta\neq 0$, $a,b>0$, and r are given real number. One has a homogeneous Dirichlet boundary conditions for $x=0$ and non-homogeneous Neumann boundary conditions for $x=1$.

For ease of presentation we take $a=b=r=1$ and derive a FE approximation as follows

1. Define the space $V=\{v\colon [0,1]\to\mathbb{R}:\ v,v'\in L^2(0,1), v(0)=0\}$. Multiply the DE with a test function $v\in V$, integrate over the domain $[0,1]$ and get the VF
$$\text{Find } u\in V \quad \text{such that} \quad (u',v')_{L^2}+(u',v)_{L^2}=\int_0^1 v(x)\,\mathrm{d}x+\beta v(1) \quad \forall v\in V.$$
2. Next, define the finite dimensional space $V_h=\{v\colon [0,1]\to\mathbb{R}:\ v \text{ is cont. pw. linear on } T_h, v(0)=0\}$, where as before T_h is a uniform partition with mesh $h=\frac{1}{m+1}$.
Observe that $V_h=\operatorname{span}(\varphi_1,\ldots,\varphi_m,\varphi_{m+1})\subset V$, with the hat functions φ_j.
The FE problem then reads
$$\text{Find } u_h\in V_h \quad \text{such that} \quad (u_h',v_h')_{L^2}+(u_h',v_h)_{L^2}=\int_0^1 v_h(x)\,\mathrm{d}x+\beta v_h(1) \quad \forall v_h\in V_h.$$

3. Choosing $v_h = \varphi_i$, writing $u_h(x) = \sum_{j=1}^{m+1} \zeta_j \varphi_j(x)$, observing that φ_{m+1} is a half hat function, and inserting everything into the FE problem gives the following linear system of equations
$$(S+C)\zeta = b,$$
where the $(m+1)\times(m+1)$ stiffness matrix S has entries $s_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x$, the $(m+1)\times(m+1)$ convection matrix C has entries $c_{ij} = \int_0^1 \varphi_j'(x)\varphi_i(x)\,\mathrm{d}x$, and the $(m+1)\times 1$ vector b has entries $b_i = \int_0^1 \varphi_i(x)\,\mathrm{d}x + \beta\varphi_i(1)$. Detailed formulas for these entries can be found in the book (Section 5.3). Solving this system gives the vector ζ and in turns the FE approximation u_h.

- Let $f\colon (0,1)\to\mathbb{R}$ be bounded and continuous. Then, the BVP
$$\begin{cases} -u''(x) = f(x) & \text{for} \quad x\in(0,1) \\ u(0)=0 \quad \text{and} & u(1)=0 \end{cases}$$
is equivalent to the VF
$$\text{Find} \quad u\in\mathscr{C}^2(0,1)\cap H_0^1 \quad \text{such that} \quad (u',v')_{L^2(0,1)} = (f,v)_{L^2(0,1)} \quad \text{for all} \quad v\in H_0^1.$$

- Poincaré inequality reads: Let $L>0$ and consider the open interval $\Omega=(0,L)$. Assume that $u\in H_0^1(\Omega) = \{v\colon \Omega\to\mathbb{R}:\ v,v'\in L^2(\Omega), v(0)=v(L)=0\}$. Then, one has
$$\|u\|_{L^2(\Omega)} \leq C_L \left\|u'\right\|_{L^2(\Omega)}.$$

- A priori error estimate in the energy norm. Let $f\colon (0,1)\to\mathbb{R}$ be bounded and continuous. Consider the BVP
$$\begin{cases} -u''(x) = f(x) & \text{for} \quad x\in(0,1) \\ u(0)=0 \quad \text{and} & u(1)=0. \end{cases}$$
Denote by u_h the solution to the corresponding FE problem (cG(1) FE). Assume that $u\in\mathscr{C}^2(0,1)$. Then, there exists a $C>0$ such that
$$\|u-u_h\|_E \leq Ch\left\|u''\right\|_{L^2(0,1)},$$
where $\|v\|_E = \sqrt{(v,v)_E} = \sqrt{(v',v')_{L^2(0,1)}}$ denotes the energy norm.

- For indication, and for a uniform partition of $[0,1]$ denoted by T_h: $x_0=0<x_1<x_2<\ldots<x_m<x_{m+1}=1$ with element length/mesh denoted by h, we summarise the possible choices for the FE spaces:
  1. Dirichlet BC $u(0)=0, u(1)=0$: test and trial spaces given by $span(\varphi_1,\ldots,\varphi_m)$.
  2. Dirichlet BC $u(0)=\alpha\neq 0, u(1)=0$: trial given by $span(\varphi_0,\varphi_1,\ldots,\varphi_m)$ and test by $span(\varphi_1,\ldots,\varphi_m)$.
  3. Dirichlet BC $u(0)=0, u(1)=\beta\neq 0$: trial given by $span(\varphi_1,\ldots,\varphi_m,\varphi_{m+1})$ and test by $span(\varphi_1,\ldots,\varphi_m)$.
  4. Dirichlet BC $u(0)=\alpha\neq 0, u(1)=\beta\neq 0$: trial given by $span(\varphi_0,\varphi_1,\ldots,\varphi_{m+1})$ and test by $span(\varphi_1,\ldots,\varphi_m)$.
  5. Dirichlet/Neumann BC $u(0)=0, u'(1)=\beta$ (zero or not): trial given by $span(\varphi_1,\ldots,\varphi_{m+1})$ and test by $span(\varphi_1,\ldots,\varphi_{m+1})$.

6. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = 0$: trial given by $span(\varphi_0, \dots, \varphi_m)$ and test by $span(\varphi_0, \dots, \varphi_m)$.
7. Dirichlet/Neumann BC $u(0) = \alpha \neq 0, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_1, \dots, \varphi_{m+1})$.
8. Neumann/Dirichlet BC $u'(0) = \alpha$ (zero or not), $u(1) = \beta \neq 0$: trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_0, \dots, \varphi_m)$.
9. Neumann BC $u'(0) = \alpha, u'(1) = \beta$ (zero or not): trial given by $span(\varphi_0, \dots, \varphi_{m+1})$ and test by $span(\varphi_0, \dots, \varphi_{m+1})$.

Further resources:

- https://web.stanford.edu/class/energy281/FiniteElementMethod.pdf
- http://mitran-lab.amath.unc.edu/courses/MATH762/bibliography/LinTextBook/chap6.pdf
- https://www.youtube.com/watch?v=WwgrAH-IMOk&ab_channel=SeriousScience (good!)
- www.simscale.com
- wiki
- wiki
- cs.uchicago.edu
- youtube