Chapter 5: Scalar initial value problems (summary)

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Goal: We study and approximate numerically solutions to IVP

$$\begin{cases} \dot{y}(t) = f(y(t)) & \text{for} \quad t \in (0,T] \\ y(0) = y_0. \end{cases}$$

Here, T, y_0, f are given and $\dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(t)$.

- Let $f, a\colon [0,T] \to \mathbb{R}$ be continuous with $a \geq 0$ bounded for instance. Let $y_0 \in \mathbb{R}$. Consider the first order linear DE

  $$\begin{cases} \dot{y}(t) + a(t)y(t) = f(t) & \text{for} \quad t \in (0,T] \\ y(0) = y_0. \end{cases}$$

  The exact solution to the above IVP is given by the variation of constants formula (voc)

  $$y(t) = y_0 \mathrm{e}^{-A(t)} + \int_0^t \mathrm{e}^{-(A(t)-A(s))} f(s)\,\mathrm{d}s,$$

  where $A(t) = \displaystyle\int_0^t a(s)\,\mathrm{d}s$.

  If $a(t) \geq 0$ for all $t \in (0,T]$ (parabolic case), we have the following stability estimate

  $$|y(t)| \leq |y_0| + \int_0^t |f(s)|\,\mathrm{d}s.$$

  If $a(t) \geq \alpha > 0$ for all $t \in (0,T]$ (dissipative case), we have the following stability estimate

  $$|y(t)| \leq \mathrm{e}^{-\alpha t}|y_0| + \frac{1}{\alpha}(1-\mathrm{e}^{-\alpha t}) \max_{0 \leq s \leq T} |f(s)|.$$

- For $y\colon \mathbb{R} \to \mathbb{R}$ differentiable at t_0 and a fixed (small) $h > 0$, we define the following approximations of the derivative:

  The forward difference

  $$\dot{y}(t_0) \approx \frac{y(t_0+h) - y(t_0)}{h}.$$

  The backward difference

  $$\dot{y}(t_0) \approx \frac{y(t_0) - y(t_0-h)}{h}.$$

  The central difference or centered difference

  $$\dot{y}(t_0) \approx \frac{y(t_0+h) - y(t_0-h)}{2h}.$$

- Consider the IVP

  $$\begin{cases} \dot{y}(t) = f(y(t)) & \text{for} \quad t \in (0,T] \\ y(0) = y_0. \end{cases}$$

  Let $N \in \mathbb{N}$ and define the time step $k = \frac{T}{N}$ as well as the grid $0 = t_0 < t_1 < \ldots < t_N = T$, where $t_n = nk$ for $n = 0, 1, \ldots, N$.

We define the following time integrators for the numerical approximations of solutions to the above IVP (starting with $y_0 = y(0)$):

The (forward/explicit) Euler scheme

$$y_{n+1} = y_n + kf(y_n).$$

The backward/implicit Euler scheme

$$y_{n+1} = y_n + kf(y_{n+1}).$$

The Crank–Nicolson scheme

$$y_{n+1} = y_n + \frac{k}{2}\big(f(y_n) + f(y_{n+1})\big).$$

These provide numerical approximations $y_n \approx y(t_n)$ to the exact solution of the IVP on the time grid $(t_n)_{n=0}^N$.

- One can apply the above time integrators to the following system of linear DE

$$\begin{cases} M\dot{\zeta}(t) + S\zeta(t) = F(t) & \text{for} \quad t \in (0,T] \\ \zeta(0) \quad \text{given} \end{cases}$$

resulting from a full discretisation of the heat equation, see the computer lab and the next chapter. In this case, M would be a mass matrix, S a stiffness matrix, and F would correspond to the load vector.

The Euler scheme then reads (with $\zeta^{(0)} = \zeta(0)$)

$$M\zeta^{(n+1)} = (M - kS)\zeta^{(n)} + kF(t_n).$$

The backward Euler scheme then reads (with $\zeta^{(0)} = \zeta(0)$)

$$(M + kS)\zeta^{(n+1)} = M\zeta^{(n)} + kF(t_{n+1}).$$

The Crank–Nicolson scheme then reads (with $\zeta^{(0)} = \zeta(0)$)

$$\left(M + \frac{k}{2}S\right)\zeta^{(n+1)} = \left(M - \frac{k}{2}S\right)\zeta^{(n)} + \frac{k}{2}\left(F(t_n) + F(t_{n+1})\right).$$

Observe that, for each of the above numerical schemes, one has to solve a linear system at each time step.

Further resources:

- In the film Hidden Figures, Katherine Goble resorts to the Euler method in calculating the re-entry of astronaut John Glenn from Earth orbit link
- www.wikipedia.org
- brown.edu
- ocw.mit.edu
- math.lamar.edu
- calcworkshop.com
- intmath.com