Chapter 6: PDE in 1d (summary)

November 25, 2022

Goal: Use FEM and the time integrators from the previous chapters to numerically discretise the heat and wave equations in $1d$.

- Consider the inhomogeneous heat equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) \quad 0 < x < 1, 0 < t \leq T \\ u(0,t) = u(1,t) = 0 \quad 0 < t \leq T \\ u(x,0) = u_0(x) \quad 0 < x < 1, \end{cases}$$

  where u_0 and f are given functions.

  Since it is seldom possible to find the exact solution $u(x,t)$ to the above problem, we need to find a numerical approximation of it. We proceed as follows

  1. To get a VF of the heat equation, consider the test/trial space $H_0^1 = \{v\colon [0,1] \to \mathbb{R}:\ v, v' \in L^2(0,1), v(0) = v(1) = 0\}$. Then, multiply the DE by a test function $v \in H_0^1$, integrate over $[0,1]$, and use integration by parts to get the VF:
     For all $0 < t \leq T$

$$\text{Find} \quad u(\cdot,t) \in H_0^1 \quad \text{s.t.} \quad (u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = \big(f(\cdot,t), v\big)_{L^2} \quad \forall v \in H_0^1 \tag{VF}$$

     with the initial condition $u(x,0) = u_0(x)$.

  2. To get a FE problem, we consider the following subspace of the above space H_0^1
     $V_h^0 = \{v_h\colon [0,1] \to \mathbb{R}:\ v_h \text{ cont. pw. linear on unif. partition } T_h, v_h(0) = v_h(1) = 0\} = \text{span}(\varphi_1, \ldots, \varphi_m)$, where $h = \frac{1}{m+1}$ and φ_j are the hat functions.
     The FE problem then reads:
     For all $0 < t \leq T$

$$\text{Find} \quad u_h(\cdot,t) \in V_h^0 \quad \text{s.t.} \quad \big(u_{h,t}(\cdot,t), v_h\big)_{L^2} + \big(u_{h,x}(\cdot,t), v_{h,x}\big)_{L^2} = \big(f(\cdot,t), v_h\big)_{L^2} \quad \forall v_h \in V_h^0 \tag{FE}$$

     with the initial condition $u_h(x,0) = \Pi_h u_0(x)$ the cont. pw. linear interpolant of u_0.

  3. From the above FE problem, we obtain a system of linear ODE by choosing the test functions $v_h = \varphi_i$ for $i = 1, \ldots, m$ and writing $u_h(x,t) = \sum_{j=1}^m \zeta_j(t)\varphi_j(x)$ with unknown coordinates $\zeta_j(t)$.
     Inserting everything in (FE), one gets the ODE

$$\begin{aligned} M\dot{\zeta}(t) + S\zeta(t) &= F(t) \\ &\zeta(0), \end{aligned} \tag{ODE}$$

     where M is the (already seen) $m \times m$ mass matrix, S is the (already seen) $m \times m$ stiffness matrix, $F(t)$ is an $m \times 1$ vector with entries $F_i(t) = (f(\cdot,t), \varphi_i)_{L^2}$ for $i = 1, \ldots, m$, the initial condition is given by

$$\zeta(0) = \begin{pmatrix} u_0(x_1) \\ \vdots \\ u_0(x_m) \end{pmatrix},$$

and the unknown vector reads

$$\zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \vdots \\ \zeta_m(t) \end{pmatrix}.$$

4. To find a numerical approximation of $\zeta(t)$ at some discrete time grid $t_0 = 0 < t_1 < \ldots < t_N = T$, with $t_j - t_{j-1} = k = \frac{T}{N}$, one can for instance use backward Euler scheme which reads

$$\begin{aligned} \zeta^{(0)} &= \zeta(0) \\ (M + kS)\zeta^{(n+1)} &= M\zeta^{(n)} + kF(t_{n+1}) \quad \text{for} \quad n = 0, 1, 2, \ldots, N-1. \end{aligned}$$

Solving these linear systems at each time step provides numerical approximations $\zeta^{(n)} \approx \zeta(t_n)$ that can be inserted in the FE solution to get approximations to the exact solution to the heat equation $u_h^k(x, t_n) = \sum_{j=1}^m \zeta_j^{(n)} \varphi_j(x) \approx u(x, t_n)$.

- Consider the wave equation (inhomogeneous) wave equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t) & 0 < x < 1, 0 < t \leq T \\ u(0,t) = u(1,t) = 0 & 0 < t \leq T \\ u(x,0) = u_0(x) & 0 < x < 1, \\ u_t(x,0) = v_0(x) & 0 < x < 1, \end{cases}$$

where u_0, v_0 and f are given functions.

Introducing a new variable for the velocity $v = u_t$, one can rewrite the above wave equation as a system of first order

$$w_t(x,t) = Aw(x,t) + F(x,t),$$

with $w(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, $F(x,t) = \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix}$ and the operator $A = \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$.

For the homogeneous wave equation, that is when $f \equiv 0$, one has conservation of the energy

$$\frac{1}{2}\|u_t(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|u_x(\cdot,t)\|_{L^2}^2 = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{1}{2}\|u_0'\|_{L^2}^2.$$

The discretisation of the wave equation is similar to the one seen above for the heat equation (we use the same notation as above):

1. The VF reads: Find $u(\cdot,t) \in H_0^1$, for all $0 < t \leq T$, such that

$$(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2}$$

for all test functions $v \in H_0^1$ and with initial conditions $u(x,0) = u_0(x)$, $u_t(x,0) = v_0(x)$.

2. The FE problem reads: Find $u_h(\cdot,t) \in V_h^0$, for all $0 < t \leq T$, such that

$$(u_{h,tt}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_{h,x})_{L^2} = (f(\cdot,t), v_h)_{L^2}$$

for all test functions $v_h \in V_h^0$ and initial conditions $u_h(x,0) = \Pi_h u_0(x)$, $u_{h,t}(x,0) = \Pi_h v_0(x)$.

3. The linear system of ODEs is given by

$$M\dot{\zeta}(t) = M\eta(t)$$
$$M\dot{\eta}(t) + S\zeta(t) = F(t).$$

Finally, one obtains a numerical approximation of the solution to this ODE by using the Crank–Nicolson scheme with time step k for instance:

$$\begin{pmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M \end{pmatrix} \begin{pmatrix} \zeta^{(n+1)} \\ \eta^{(n+1)} \end{pmatrix} = \begin{pmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M \end{pmatrix} \begin{pmatrix} \zeta^{(n)} \\ \eta^{(n)} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k}{2}\left(F(t_{n+1}) + F(t_n)\right) \end{pmatrix}.$$

The above provides an approximation $u_h^k(t_n, x) = \sum_{j=1}^{m} \zeta_j^{(n)} \varphi_j(x)$ of the solution u of the above wave equation.

Further resources:

- www.wikipedia.org
- math.lamar.edu
- www.wikipedia.org
- www.brilliant.org
- math.lamar.edu
- chem.libretexts.org