Chapter 9: Partial differential equations (summary)

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Goal: Present a tool to find exact solutions to particular PDEs.

- Consider partial differential equations of the form

$$\mathscr{L}(u) = F,$$

where $\mathscr{L}$ is a linear differential operator (i. e. $\mathscr{L}(c_1u_1 + c_2u_2) = c_1\mathscr{L}(u_1) + c_2\mathscr{L}(u_2)$ for any real constants c_1, c_2) and $F = F(x)$ can be regarded as an external force for example. This abstract equation is called homogeneous if $F \equiv 0$ and inhomogeneous if $F \neq 0$. Furthermore, one has to add, to the above problem, linear boundary conditions of the form (think Dirichlet or Neumann boundary conditions)

$$B(u) = f \quad \text{on the boundary},$$

where B is again a linear operator and $f = f(x)$. We have homogeneous boundary conditions if $f \equiv 0$, inhomogeneous boundary conditions otherwise.

Example: The abstract form of the one-dimensional linear inhomogeneous heat equation on the interval $[0,1]$ is

$$\begin{cases} u_t(x,t) = 3u_{xx}(x,t) + x, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \sin(x) \end{cases}$$

is given by taking $\mathscr{L}(u) = u_t - 3u_{xx}, F(x) = x$, $B(u) = u$ on the boundary, $f(x) = 0$ and the initial condition $u(x,0) = \sin(x)$.

- The superposition principle can be used to solve smaller or easier problems from a more complicated original problem. We have the following: If $u_1, u_2, \ldots, u_k$ satisfy the linear PDEs

$$\begin{aligned} \mathscr{L}(u_j) &= F_j \\ B(u_j) &= f_j \quad j = 1, 2, \ldots, k \end{aligned}$$

and $c_1, c_2, \ldots, c_k$ are any constants, then $u := c_1u_1 + c_2u_2 + \ldots + c_ku_k$ satisfies the more complicated problem

$$\begin{aligned} \mathscr{L}(u) &= c_1F_1 + c_2F_2 + \ldots + c_kF_k \\ B(u) &= c_1f_1 + c_2f_2 + \ldots + c_kf_k. \end{aligned}$$

The above tells us, for instance, that to solve the inhomogeneous PDE $\mathscr{L}(u) = F, B(u) = f$ it is enough to find the solution, denoted u_H, to the homogeneous PDE $\mathscr{L}(u) = 0, B(u) = 0$ and one particular solution, denoted u_p, to the problem $\mathscr{L}(u) = F, B(u) = f$. The general solution will then be given by $u = u_H + u_p$.

- The technique of separation of variables can be used to solve homogeneous linear partial differential equations $L(u) = 0, B(u) = 0$ with initial conditions if needed. For the heat or wave equation, the main idea is to set $u(x,t) = X(x)T(t)$ and find two ordinary differential equations for the functions X and T. If one can solve these differential equations (using in addition the given boundary conditions), one finds solutions to the original PDE (here, one can use Fourier series and the superposition principle). Observe that this technique works only for particular problems. As application, we have used separation of variables to find exact solutions to the homogeneous heat and wave equations.

Combining the above with the superposition principle, we were also able to find the exact solution to some inhomogeneous PDEs like the heat equation.

- As a final application, we have seen how to solve the transport equation (where $\alpha > 0$ and $C > 0$ are given)

$$\begin{cases} u_t(x,t) = -\alpha u_x(x,t), & \text{for } x, t > 0 \\ u(0,t) = C, \\ u(x,0) = 0 \end{cases}$$

using the Laplace transform.

The main steps are the same as in Chapter 7: Take the Laplace transform (in t) of the problem. Solve an ODE. Take the inverse Laplace transform (in s) in order to get the solution $u(x,t)$.

Further resources:

- wikipedia.org (PDE, linear, sep. of variables, superposition)
- wikibooks.org (linear PDE)
- ocw.mit.edu (heat eq.)
- ocw.mit.edu (wave eq.)
- math.etsu.edu (PDE, sep. of variables)
- www.ucl.ac.uk (PDE, sep. of variables)