Exam, 14.01.23, TMA683

(1) No! not linear since $(\ldots)^3$ and not of order 3 (no u_{xxx})

(2) No since one can have $\langle A,A\rangle = 0$ with $A \neq \begin{pmatrix}0&0\\0&0\end{pmatrix}$. Ex. $A = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

(3) Formula for line between 2 points: $y = x+1$ ($\exists!$ interpolant)

(4) Area $= \int_0^1 x^2\,dx \approx \frac{(1-0)}{2}(0^2+1^2) = \frac{1}{2}$

(5) Exact sol. to ODE: $y(t) = e^{3t}$

Let $N \in \mathbb{N}$, set $k = \frac{1}{N}$ for stepsize and $t_n = n\cdot k$ for $n = 0,1,2,\ldots$

Euler: $y_1 = y_0 + k f(y_0) = 1 + 3k \to \ldots \to y_n = (1+3k)^n \approx y(t_n) = e^{3t_n}$

(6) Hat fct have disjoint support $\Rightarrow$ mass/stiffness matrix has a lot of zeros

(7) a) Hom. Dirichlet BC $\Rightarrow$ test = trial = $H_0^1(\Omega)$

(VF) Find $u \in H_0^1$ s.t. $(u_x, v_x)_{L^2} + 4(u,v)_{L^2} = (x,v)_{L^2}$ $\forall v \in H_0^1$

b) FE sol. $u_h(x) = \sum_{j=1}^3 \xi_j \varphi_j(x) = \sum_{j=1}^3 \xi_j \sin(jx)$, test fct $v_i(x) = \sin(ix)$ for $i = 1,2,3$.

Into FE prob. gives $\sum_{j=1}^3 ij\,(\cos(jx), \cos(ix))_{L^2}\xi_j + 4\sum_{j=1}^3 (\sin(jx), \sin(ix))_{L^2}\xi_j = (x, \sin(ix))$

$i = 1,2,3$

So $(K + 4M)\xi = F$ where $K = \left(ij\int_0^\pi \cos(jx)\cos(ix)\,dx\right)_{i,j=1}^3$

$M = \left(\int_0^\pi \sin(jx)\sin(ix)\,dx\right)_{i,j=1}^3$

$F = \left(\int_0^\pi x\sin(ix)\,dx\right)_{i=1}^3 = \begin{pmatrix}\pi\\ -\pi/2\\ \pi/3\end{pmatrix}$

(8) a) $\|u-u_h\|_{L^2}^2 = \int_0^1 (u-u_h)\underbrace{(u-u_h)}_{\in H_0^1}\,dx \overset{(Aux)}{=} \int_0^1 \zeta'(u'-u_h')\,dx \overset{(GO)}{=} \int_0^1 (\zeta - \pi_h\zeta)'(u'-u_h')\,dx$

$\pi_h\zeta \in V_h^0$

b) $\|u-u_h\|_{L^2}^2 = \int_0^1 (\zeta-\pi_h\zeta)'(u-u_h)'\,dx \overset{C-S}{\leq} \|(\zeta-\pi_h\zeta)'\|_{L^2}\|(u-u_h)'\|_{L^2} \leq$

$\underset{\text{interpol. error}}{\leq} C\cdot h\cdot\|\zeta''\|_{L^2}\|(u-u_h)'\|_{L^2} \leq C\cdot h\cdot\|u-u_h\|_{L^2}\|(u-u_h)'\|_{L^2}$

(9) Write $\frac{3s^2-28}{(s-4)(s^2+4)} = \frac{A}{s-4} + \frac{Bs+C}{s^2+4}$ $\Rightarrow \ldots \Rightarrow$ get $A = 1, B = 2, C = 8$

Thus, $\mathcal{L}^{-1}\left\{\frac{3s^2-28}{(s-4)(s^2+4)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}(t) + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}(t) + 8\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}(t) =$ 1

$\underset{\text{Table}}{=} e^{4t} + 2\cos(2t) + 4\sin(2t)$ 1

10) Set $Y(s) = \mathcal{L}\{y(t)\}(s)$

Take LT of prob: $sY(s) - y(0) + 2Y(s) = \frac{1}{s} - e^{-s}\frac{1}{s}$ (table) 2

Solve for $Y(s)$: $Y(s) = \frac{1}{s(s+2)} - e^{-s}\frac{1}{s(s+2)} - 2\frac{1}{s+2}$ 1

Take ILT: $y(t) = \frac{1}{2}(1-e^{-2t}) - \frac{1}{2}(1-e^{-2(t-1)})\theta(t-1) - 2e^{-2t}$ 1

11) f periodic if $f(x+p) = f(x)\ \forall x$ 1

12) $c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ 1

13) $f \to$ odd 1 $\to a_n = 0$ for $n = 0, 1, 2, \dots$ 1

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$ 1 $= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \dots = \frac{2}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + 0\right) = -\frac{2}{n}(-1)^n$ 1

for $n = 1, 2, \dots$

14) $f(x) = x^2 = \underbrace{\frac{\pi^2}{3}}_{a_0/2} + 4\cdot\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx)$

(i) Take $x = \pi$ 1 $\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}(-1)^k$ 1 $\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$ 1

(ii) Parseval $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{4}{2}\cdot\frac{\pi^4}{9} + \sum_{k=1}^{\infty}\frac{16}{k^4}$ 1 $\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$ 1

0.5

15) Set $u(x,y) = X(x)Y(y)$ into prob $\to x\frac{X'(x)}{X(x)} = 1 + y\frac{Y'(y)}{Y(y)} = k$ (constant) 0.5

1 $xX'(x) = kX(x) \Leftrightarrow \int\frac{1}{X(x)}\,dX = \int\frac{k}{x}\,dx \Rightarrow X(x) = Cx^k$ and $Y(y) = Ay^{k-1}$ + BC gives $u(x,y) = 8x^3y^2$ 1

1 $yY'(y) = Y(y)(k-1)$

16) (BVP) $\begin{cases} S''(x) = 6x \\ S(0) = S(1) = 0 \end{cases}$ 1 (PDE) $\begin{cases} v_t - v_{xx} = 0 \\ v(0,t) = 0,\ v(1,t) = 0 \\ v(x,0) = f(x) - S(x) \end{cases}$ 1

$S(x) = Ax^3 + Bx^2 + Cx + D \rightsquigarrow \dots \to S(x) = x^3 - x$ 1