Examination, 03 April 2023
TMA683

Read this before you start!
I'll try to come at ca. 10:00. You can ask for calling me (0317723021) in case of questions.
Aid: Personal pocket calculator.
A table of Laplace transforms is provided at the end of the exam.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
*Use a proper pen and order your answers if possible.**Thank you!***
No need to use one piece of paper for only one exercise.
The test has 4 pages and a total of 50 points.
Preliminary grading limits: 3: 20-29p, 4: 30-39p and 5: 40-50p.
Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.

Good luck!

Some exercises were taken from, or inspired by, materials from *P. Dawkins*, *P. Frey*.

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1. Provide concise answers (with justifications) to the following short questions:

   (a) Let an integer $q > 0$. What is the dimension of the polynomial space $\mathcal{P}^{(q)}(0,1)$? Give an application of such spaces. (2p)

   (b) Consider a uniform partition of $[0,1]$ with mesh h. What is the size (in term of h) of the error of the continuous piecewise linear interpolant? (1p)

   (c) Use (one step of) the midpoint rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (2p)

   (d) Give the general formula for the Crank–Nicolson scheme, with time step k, when applied to the IVP $\dot{y}(t) = f(y(t)), y(0) = y_0$. (2p)

   (e) Is the variational formulation and the strong formulation of Poisson's equation in $1d$ with homogeneous Dirichlet BC equivalent, assuming that $u \in C^2$? (1p)

   (f) State Poincaré's inequality on $\Omega = (0,1)$ (the value of the constant is not important). (2p)

   (g) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size h for the model problem (Poisson's equation in $1d$ with homogeneous Dirichlet BC) seen in the lecture? (2p)

(h) In which norm is the cG(1) approximation of Poisson's equation in $1d$ with homogeneous Dirichlet BC the best approximation of the solution to this BVP in the finite element space? (1p)

(i) Is the energy of the homogeneous linear heat equation with homogeneous Dirichlet BC in general a conserved quantity? (1p)

2. Consider the following pseudo-code

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function [t,y] = solveit(t0, y0, T, n)
  t = zeros(n+1,1); y = zeros(n+1,1);
  t(1) = t0; y(1) = y0; h = (T-t0)/n;
  for i = 1:n
    t(i+1) = t(i)+h;
    y(i+1) = y(i)+h*(y(i))^2;
  end
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Suppose that the input values are $t0 = 0, y0 = 1, T = 1$, and $n = 10$.

What is the initial-value problem being approximated numerically? What is the numerical method being used? (2p)

3. Let $f\colon (0,1) \to \mathbb{R}$, $\alpha \neq 0$ and $\beta \neq 0$. Let us apply cG(1) to the BVP

$$\begin{cases} -u'' = f & \text{in } \Omega = (0,1) \\ u(0) = \alpha, u(1) = \beta. \end{cases}$$

(a) Derive the variational formulation of this BVP (define all quantities). (2p)

(b) Derive the corresponding finite element problem (define all quantities). (2p)

(c) Give the linear system of equations for the computation of the finite element solution. You don't need to compute the values of the integrals in the stiffness matrix (a general formula is enough) but you must give all coordinates of the load vector. (4p)

4. Let $\Omega = (0,1)$ and $f \in L^2(\Omega)$. Consider the problem

Find $u \in H_0^1(\Omega)$ such that $\int_\Omega u'(x)v'(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

Let $u_h \in V_h^0$ be the corresponding cG(1) approximation to u on a uniform partition with mesh size h. Consider then the auxiliary problem

Find $\zeta \in H_0^1(\Omega)$ such that $\int_\Omega \zeta'(x)v'(x)\,\mathrm{d}x = \int_\Omega (u(x) - u_h(x))\, v(x)\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

(a) Show Galerkin's orthogonality

$$\int_\Omega (u(x) - u_h(x))' v_h'(x)\,\mathrm{d}x = 0 \quad \text{for all } v_h \in V_h^0. \qquad (2p)$$

(b) Using the above auxiliary problem and Galerkin's orthogonality, show that

$$\|u-u_h\|^2_{L^2(\Omega)} = \int_\Omega (u(x)-u_h(x))'(\zeta(x)-\pi_h\zeta(x))'\,\mathrm{d}x,$$

where we recall that $\pi_h\zeta \in V_h^0$ denotes the continuous piecewise linear interpolant of ζ. (3p)

(c) Next, using an interpolation error estimate from the lecture (observe that $\zeta \in H^2(\Omega)$ since $-\zeta'' = u-u_h$), show the following error estimate for the cG(1) approximation:

$$\|u-u_h\|_{L^2(\Omega)} \leq Ch\,\|(u-u_h)'\|_{L^2(\Omega)}\,. \qquad (2p)$$

5. Is the function $f(t)=e^{-5t}$, for $t \geq 0$, of exponential order α and piecewise continuous? How can you guarantee the existence of the Laplace transform of f? (3p)

   Hint: Don't forget to recall the definitions of exponential order and piecewise continuous.

6. Compute the Laplace transform of

$$f(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t). \qquad (3p)$$

7. Is the inverse Laplace transform a linear map? Justify. (2p)

8. Find the inverse Laplace transform of

$$F(s) = \frac{6}{s} + \frac{19}{s+2} + \frac{8}{3s^2+12}. \qquad (3p)$$

9. Let f be a periodic function of period p. How do you write the Fourier coefficient a_0 of the Fourier series of f? (1p)

10. Find the Fourier series of $f(x)=x$ defined on $-\pi \leq x \leq \pi$ and extended periodically. (4p)

11. Consider the nonhomogeneous PDE ($0<x<1$, $0<t<T$)

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = -6x \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x), \end{cases}$$

with a given (nice) function f. Using the superposition principle and the ansatz $u(x,t)=v(x,t)+s(x)$, find a homogeneous PDE for v (no need to solve this problem) and solve the corresponding BVP for s. (3p)

Table of Laplace Transforms and trigonometry

$f(t)$	$F(s)$
$af(t)+bg(t), \quad a,b\in\mathbb{R}$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t). \quad n=1,2,\ldots$	$(-1)^n F^{(n)}(s)$
$\mathrm{e}^{-at}f(t), \quad a\in\mathbb{R}$	$F(s+a)$
$f(t-T)\theta(t-T), \quad T\in\mathbb{R}$	$\mathrm{e}^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t), \quad n=1,2,\ldots$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,\mathrm{d}\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}, \quad s>0$
$\frac{t^n}{n!}, \quad n=1,2,\ldots$	$\frac{1}{s^{n+1}}, \quad s>0$
$\mathrm{e}^{-at}, \quad a\in\mathbb{R}$	$\frac{1}{s+a}, \quad s>-a$
$\cosh(at), \quad a\in\mathbb{R}$	$\frac{s}{s^2-a^2}, \quad s>\|a\|$
$\sinh(at), \quad a\in\mathbb{R}$	$\frac{a}{s^2-a^2}, \quad s>\|a\|$
$\cos(bt), \quad b\in\mathbb{R}$	$\frac{s}{s^2+b^2}, \quad s>0$
$\sin(bt), \quad b\in\mathbb{R}$	$\frac{b}{s^2+b^2}, \quad s>0$
$\frac{t}{2b}\sin(bt), \quad b\in\mathbb{R}\backslash\{0\}$	$\frac{s}{(s^2+b^2)^2}, \quad s>0$
$\frac{1}{2b^3}\left(\sin(bt)-bt\cos(bt)\right), \quad b\in\mathbb{R}\backslash\{0\}$	$\frac{1}{(s^2+b^2)^2}, \quad s>0$

$$2\sin(a)\sin(b)=\cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b)=\sin(a-b)+\sin(a+b)$$
$$2\cos(a)\cos(b)=\cos(a-b)+\cos(a+b)$$