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Exam 03.04.23

TMA683

1 a) $\dim(P^{(q)}(0,1)) = q+1$ 1. Applications: interpolation (f.ex) 1

b) $\|\pi_h f - f\|_{L^2} \leq C \cdot h^2 \|f''\|_{L^2}$ or $\leq C \cdot h \|f'\|_{L^2}$ 1

c) $\int_1^3 f(x)\,dx \approx (3-1)\, f\left(\frac{3+1}{2}\right) = 8$ 1 1

d) $y_{n+1} = y_n + \frac{k}{2}\left(f(y_n) + f(y_{n+1})\right)$ 1

e) Yes 1

f) $\|u\|_{L^2(\Omega)} \leq C \cdot \|u'\|_{L^2(\Omega)}$ 1 $\forall u \in H_0^1(\Omega)$ 1

g) $\|u - u_h\|_{E} \leq C \cdot h$ 1 1

h) Energy norm 1

i) No (in general for heat). Yes for wave... 1

2 See line 6, ODE $\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$ 1 Scheme: exp. Euler scheme 1

3 a) Find $u \in H^1(\Omega)$ with $u(0) = \alpha,\ u(1) = \beta$ s.t. $(u', v')_{L^2} = (f, v)_{L^2}\ \forall v \in H_0^1$ 1

b) Let $h = \frac{1}{m+1}$, $T_h: 0 = x_0 < x_1 < \dots < x_{m+1} = 1$, $V_h = \mathrm{span}(\{\varphi_j\}_{j=0}^{m+1})$, $V_h^0 = \mathrm{span}(\{\varphi_j\}_{j=1}^{m})$ 1

Find $u_h \in V_h$ s.t. $(u_h', v_h')_{L^2} = (f, v_h)_{L^2}$ $\forall v_h \in V_h^0$ 1

c) Write $u_h = \sum_{j=0}^{m+1} \xi_j \varphi_j$ where $\xi_0 = \alpha$, $\xi_{m+1} = \beta$ $\Rightarrow$ m unknown ξ_j 1

Take $v_h = \varphi_i$ for $i = 1, \dots, m$ and get $S\xi = b$, where $S = (s_{ij})_{i,j=1}^{m}$, $b = (b_i)_{i=1}^{m}$ 1

and $s_{ij} = (\varphi_j', \varphi_i')_{L^2}$ and $b_i = (f, \varphi_i)_{L^2}$ for $i = 2, \dots, m-1$ and 2

$b_1 = (f, \varphi_1)_{L^2} - \alpha(\varphi_0', \varphi_1')_{L^2}$ and $b_m = (f, \varphi_m) - \beta(\varphi_{m+1}', \varphi_m')_{L^2}$

4 a) Take $v = v_h \in V_h^0 \subset H_0^1$ into (VF) and substract (FE) to get (GO) 2

b) $\|u - u_h\|_{L^2}^2 \overset{\text{Def}}{=} \int_\Omega (u - u_h)^2\,dx \overset{\text{(Aux)}}{=} \int_\Omega \xi'(u' - u_h')\,dx \underset{\pi_h \xi \in V_h^0}{\overset{\text{(GO)}}{=}} \int_\Omega (\xi - \pi_h \xi)'(u' - u_h')\,dx$ 2

c) $\|u - u_h\|_{L^2}^2 \overset{\text{C-S}}{\leq} \|(\xi - \pi_h \xi)'\|_{L^2} \|(u - u_h)'\|_{L^2} \underset{\text{interpolation}}{\leq} C \cdot h \|\xi''\|_{L^2} \|(u - u_h)'\|_{L^2} \leq$

$\underset{\text{def } \xi''}{\leq} C \cdot h \|u - u_h\|_{L^2} \|(u - u_h)'\|_{L^2}$ 2

5 pwc: finite nbr of discontinuity and ~~limit~~ limit to left/right $\exists$ 1

exp. order: $|f(t)| \leq M \cdot e^{\alpha t}$ $\forall t \geq T$ 1

$f(t) = e^{-5t}$ is cont. and of exp. order $\alpha = 5$

for small t LT $\exists$! 1

⑥ $\mathcal{L}\{f(t)\}(s) \underset{\text{linearity}}{=} \mathcal{L}\{e^{3t}\}(s) + \mathcal{L}\{\cos(6t)\}(s) - \mathcal{L}\{e^{3t}\cos(6t)\}(s) \underset{\text{Table + shift}}{=} \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{(s-3)}{(s-3)^2+36}$ ① ① ①

⑦ Yes since one has $\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\}(t) = \alpha \mathcal{L}^{-1}\{F(s)\}(t) + \beta \mathcal{L}^{-1}\{G(s)\}(t) \quad \forall \alpha, \beta \in \mathbb{R}$

⑧ $\mathcal{L}^{-1}\{F(s)\}(t) \underset{\text{linearity}}{=} 6\,\mathcal{L}^{-1}\{\frac{1}{s}\}(t) + 19\,\mathcal{L}^{-1}\{\frac{1}{s+2}\}(t) + \frac{8}{3}\,\mathcal{L}^{-1}\{\frac{1}{s^2+4}\}(t) \underset{\text{Table}}{=} 6\,\Theta(t) + 19\,e^{-2t} + \frac{4}{3}\sin(2t)$ ① ① ①

⑨ $2L$-periodic $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \xrightarrow{2L=p} a_0 = \frac{1}{p/2}\int_{-p/2}^{p/2} f(x)\,dx$ ①

⑩ $f(x)$ odd $\Rightarrow a_n = 0 \quad \forall n = 0,1,2,\dots$ ①

$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin\left(\frac{n\pi x}{\pi}\right)dx \overset{\text{by part}}{=} \dots = 2(-1)^{n+1}\frac{2}{n} \Rightarrow f(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx)$ ① ① ①

⑪ (BVP) $\begin{cases} s''(x) = 6x \\ s(0) = s(1) = 0 \end{cases}$ ①

$s(x) = Ax^3 + Bx + Cx + D \Rightarrow s(x) = x^3 - x$ ①

(PDE) $\begin{cases} v_t - v_{xx} = 0 \\ v(0,t) = 0,\ v(1,t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases}$ ①