Exam 17.08.23
TMA 683

(1) a) ODE = a differential eq. where the unknown fct. depends only on one variable

b) Linear yes, order 2 no (no second derivatives)

c) $\dim(P^{(3)}(0,1)) = 4$, Interpolation

d) $\|\pi_h f - f\|_{L^2} \leq C\cdot h^2 \|f''\|_{L^2}$ or $C\cdot h \|f'\|_{L^2}$

e) $\int_1^3 x^2\,dx \approx \frac{3-1}{2}(3^2+1^2) = 10$

f) $y_{n+1} = y_n + \frac{k}{2}\left(f(y_n) + f(y_{n+1})\right)$

g) $\|u - u_h\|_E \leq C\cdot h$

h) $|(u,v)_V| \leq \|u\|_V \cdot \|v\|_V$

(2) line 6 → IVP: $y' = y^2$, $y(t_0) = y_0$. Implicit/backward Euler scheme

(3) Space H^1 → Neumann BC → by parts (1) (0.5) BVP reads $-u''(x) + c(x)u(x) = f(x)$ in Ω, $u'(0) = N$, $u'(1) = \beta$ (0.5)

(4) a) Dirichlet BC → test = trial = $H_0^1(\Omega)$ (1)

Multiply by test fcts, by parts, get

(VF) Find $u \in H_0^1$ s.t. $a(u',v')_{L^2} + \alpha(u,v)_{L^2} = (f,v)_{L^2}$ $\forall v \in H_0^1$ (1)

b) $V_h^0 = \{v_h \in H_0^1$ s.t. v_h cont on $[0,1]$ and linear on $[x_n, x_{n+1}]$ $\forall n$ and $v_h(0) = v_h(1) = 0\}$ (1)

(FE) Find $u_h \in V_h^0$ s.t. $a(u_h', v_h')_{L^2} + \alpha(u_h, v_h)_{L^2} = (f, v_h)_{L^2}$ $\forall v_h \in V_h^0$ (1)

c) $V_h^0 = \mathrm{Span}(\{\varphi_j\}_{j=1}^N)$ with hat fcts. and then $u_h(x) = \sum_{j=1}^N \xi_j \varphi_j(x)$, unknowns ξ_j

d) Take $v_h = \varphi_j$ for $j = 1,..,N$ into FE and get

$S\xi + M\xi = F$ (1), where $S_{ij} = (\varphi_i', \varphi_j')_{L^2}$, $M_{ij} = (\varphi_i, \varphi_j)_{L^2}$, $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}$, (1+1)

$F = ((f,\varphi_j)_{L^2})_{j=1}^N$ and $(f, \varphi_i)_{L^2} = \int_0^1 1\cdot\varphi_i(x)\,dx = 1\cdot h$ (1)

(5) a) Hom. Dirichlet → $V = H_0^1$ (1):

(VF) $\forall t \in (0,T]$, Find $u(\cdot,t) \in H_0^1$ s.t. $(u_{tt}(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} + (u(\cdot,t), v)_{L^2} = (f,v)_{L^2}$ $\forall v \in V$

$u(\cdot,0) = u_0$, $u_t(\cdot,0) = v_0$ (1)

b) $V_h^0 = \mathrm{span}(\{\varphi_j\}_{j=1}^N)$ (1) with hat fct.

(VF) $\forall t \in (0,T]$, find $u_h(\cdot,t) \in V_h^0$ s.t. $(u_{h,tt}(\cdot,t), v_h)_{L^2} + (u_{h,x}(\cdot,t), v_{h,x})_{L^2} + (u_h(\cdot,t), v_h)_{L^2} = (f, v_h)_{L^2}$ $\forall v_h \in V_h^0$

$u_h(\cdot,0) = \pi_h u_0$, $u_{h,t}(\cdot,0) = \pi_h v_0$ (1)

c) See lecture → $M = ((\varphi_j, \varphi_i)_{L^2})_{i,j=1}^{N}$ 1, $S = ((\varphi_j', \varphi_i')_{L^2})_{i,j=1}^{N}$, $F = ((f, \varphi_i)_{L^2})_{i=1}^{N}$ 1

$\xi(0) = (u_0(x_j))_{j=1}^{N}$, $\dot{\xi}(0) = (v_0(x_j))_{j=1}^{N}$

(6) $f(t) = e^{t^2}$

(7) $\mathcal{L}\{f(t)\}(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} e^{3t}\,dt = \lim_{T\to\infty} \int_0^T e^{(3-s)t}\,dt = \lim_{T\to\infty} \frac{e^{(3-s)t}}{(3-s)}\Big|_0^T = \frac{1}{s-3}$ for $s>3$ 1

1 for $s>3$

(8) $\mathcal{L}\{f(t)\}(s) \underset{\text{table}}{\overset{\text{linearity}}{=}} \dots = \frac{2}{s^3} + \frac{2}{s^2+4} + \left(\frac{1}{s^2} + \frac{5}{s}\right)e^{-2s}$

1 1 1

(9) LT calculated via integrals that are linear operations

(10) $F(s) = -\frac{1}{(s+1)^2} + \frac{1}{s+1} \xrightarrow{\text{table}} f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = (-t+1)e^{-t}$

1 1 1

(11) Take LT, $Y(s) = \mathcal{L}\{y(t)\}(s)$, and get $Y(s) = \frac{3+s}{s^2+3s+2} = \frac{2}{s+1} + \frac{(-1)}{s+2}$

1 1

Take ILT: $y(t) \underset{\text{Table}}{=} 2e^{-t} - e^{-2t}$ 1

1

(12) $f(t)$ already a linear combination of sin/cos $\Rightarrow a_5 = 1, b_2 = 1$, rest $= 0$. 1

(13) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underset{\text{even}}{f(x)}\, \underset{\text{odd}}{\sin(nx)}}_{\text{odd}}\, dx = 0$ 0.5

0.5 1

(14) $f(x)$ odd $\to a_n = 0$ 1 and $b_n = \frac{2}{\pi} \int_0^\pi x \sin(nx)\,dx \overset{!}{=}$ by parts $= (-1)^{n+1}\frac{2}{n}$ 0.5

0.5

$\Rightarrow f(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx)$ 1

(15) Parseval $\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\,dx$. Can be used to compute series

0.5 like $\sum_{n=1}^{\infty} \frac{1}{n^2}$ f. ex. 0.5