Examination, 17 August 2023
TMA683

Read this before you start!
I'll try to come at ca. 15:00. You can ask for calling me (0317723021) in case of questions.
Aid: Chalmers approved calculators.
A table of Laplace transforms is provided at the end of the exam.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible. ***Thank you!***
Feel free to write more than one solution on one piece of paper (save paper).
The test has 4 pages and a total of 50 points.
Preliminary grading limits: 3: 20-29p, 4: 30-39p and 5: 40-50p.
Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.

Good luck!

Some exercises were taken from, or inspired by, materials from *A. Szepessy*, *www.bibmath.net* and *www.tutorial.math.lamar.edu*.

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1. Provide concise answers (with justifications) to the following short questions:

   (a) Give the definition of an ODE. (1p)

   (b) Is the PDE $u_t(x,t) - u_x(x,t) + 2u(x,t) = 0$ a linear PDE of order 2? Justify your answer. (2p)

   (c) What is the dimension of the polynomial space $\mathcal{P}^{(3)}(0,1)$? Give an application of the polynomial space $\mathcal{P}^{(l)}(0,1)$, for a positive integer l. (1p)

   (d) Consider a uniform partition of $[0,1]$ with mesh h. What is the size (in term of h) of the error of the continuous piecewise linear interpolant $\left\|\pi_h f - f\right\|_{L^2(0,1)}$ for a nice function f? (1p)

   (e) Use (one step of) the trapezoidal rule to approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 3$. (1p)

   (f) Give the general formula for the Crank–Nicolson scheme, with time step k, when applied to the IVP $\dot{y}(t) = f(y(t)), y(0) = y_0$. (1p)

   (g) What is an estimate for the a priori error estimate, measured in the energy norm, of the (continuous and piecewise linear) FEM with mesh size h for the model problem (Poisson's equation in $1d$ with homogeneous Dirichlet BC) seen in the lecture? (1p)

(h) Let $(V, (\cdot,\cdot))$ be an inner product space. State Cauchy–Schwarz inequality. (1p)

2. Consider the following pseudo-code

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function [t,y] = solveit(t0, y0, T, n)
  t = zeros(n+1,1); y = zeros(n+1,1);
  t(1) = t0; y(1) = y0; h = (T-t0)/n;
  for i = 1:n
    t(i+1) = t(i)+h;
    y(i+1) = y(i)+h*(y(i+1))^2;
  end
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Suppose that the input values are $t0 = 0, y0 = 1, T = 1$, and $n = 10$.

What is the initial-value problem being approximated numerically? What is the numerical method being used? (2p)

3. Let $\Omega = (0,1)$, $\alpha, \beta \in \mathbb{R}$, and $c, f\colon \Omega \to \mathbb{R}$ be nice functions. Consider the problem: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega u'(x)v'(x)\,\mathrm{d}x + \int_\Omega c(x)u(x)v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x + \beta v(1) - \alpha v(0) \quad \text{for all } v \in H^1(\Omega).$$

Formulate the (strong) problem corresponding to the above variational formulation. (2p)

4. Let $a, r > 0$ and $f \in L^2(0,1)$. Consider the BVP

$$\begin{cases} -au''(x) + ru(x) = f \quad \text{in } \Omega = (0,1) \\ u(0) = 0, u(1) = 0. \end{cases}$$

(a) Derive the variational formulation of this BVP (define all quantities). (2p)

(b) Derive the corresponding cG(1) finite element problem (define all quantities). (2p)

(c) Provide the ansatz (in terms of the hat functions) for the FE solution $u_h(x)$. (1p)

(d) Let now $a = r = 1$ and $f(x) = 1$ for $x \in (0,1)$. Give the linear system of equations for the computation of the finite element solution. You don't need to compute the integrals in the matrices (a general formula is enough) but must give all coordinates of the right-hand side vector ("load vector"). (4p)

5. Let $\Omega = (0,1)$, $f, u_0, v_0\colon \Omega \to \mathbb{R}$ be nice, $T > 0$. Consider the inhomogeneous wave equation

$$\begin{cases} u_{tt} - u_{xx} + u = f \quad \text{in } \Omega \times (0,T] \\ u = 0 \quad \text{on } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) \quad \text{in } \Omega \\ u_t(x,0) = v_0(x) \quad \text{in } \Omega. \end{cases}$$

(a) Write the variational formulation of the above problem in a suitable space V (don't forget to define this space). (2p)

(b) Formulate the finite element problem based on a cG(1) approximation on the space V_h^0 (don't forget to define this space). (2p)

(c) From the above, derive the linear system of ordinary differential equations

$$M\ddot{\zeta}(t) + S\zeta(t) + M\zeta(t) = F(t), \text{with initial value } \zeta(0), \dot{\zeta}(0).$$

Don't forget to provide the entries of the matrices M, S and of the vectors $F(t), \zeta(0), \dot{\zeta}(0)$ (you don't need to compute the integrals). (2p)

6. Give an example of a function which is not of exponential order. (1p)

7. Use the definition of the Laplace transform (not the table!) to compute the Laplace transform of

$$f(t) = e^{3t}. \qquad (2p)$$

8. Compute the Laplace transform of

$$f(t) = t^2 + \sin(2t) + (t+3)\theta(t-2), \qquad (3p)$$

where we recall that θ denotes the Heaviside function or the unit step function.

9. Why is the Laplace transform a linear map? Justify. (2p)

10. Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s+1)^2}. \qquad (3p)$$

11. Use Laplace transforms to find the exact solution to the differential equation

$$\begin{cases} y''(t) + 3y'(t) + 2y(t) = 0 \\ y(0) = 1, y'(0) = 0. \end{cases} \qquad (3p)$$

12. Find the Fourier series of the 2π-periodic function $f(x) = \sin(2x) + \cos(5x)$ defined on $[-\pi, \pi]$. (2p)

13. Let f be an even 2π-periodic function. Show that its Fourier coefficients b_n are zero for all $n \in \mathbb{N}^*$. (2p)

14. Find the Fourier series of $f(x) = x$ defined on $-\pi \leq x \leq \pi$ and extended periodically. (3p)

15. State Parseval's identity (in terms of Fourier coefficients a_n, b_n or c_n) and give an example of its possible use (no details needed, just the main idea). (1p)

Table of Laplace Transforms and trigonometry

$f(t)$	$F(s)$
$af(t)+bg(t), \quad a,b \in \mathbb{R}$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t). \quad n=1,2,\ldots$	$(-1)^n F^{(n)}(s)$
$e^{-at}f(t), \quad a \in \mathbb{R}$	$F(s+a)$
$f(t-T)\theta(t-T), \quad T \in \mathbb{R}$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t), \quad n=1,2,\ldots$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$ for $s>0$
$\frac{t^n}{n!}$ for $n=1,2,\ldots$	$\frac{1}{s^{n+1}}$ for $s>0$
$e^{-at}, \quad a \in \mathbb{R}$	$\frac{1}{s+a}$ for $s>-a$
$\cosh(at), \quad a \in \mathbb{R}$	$\frac{s}{s^2-a^2}$ for $s>\lvert a\rvert$
$\sinh(at), \quad a \in \mathbb{R}$	$\frac{a}{s^2-a^2}$ for $s>\lvert a\rvert$
$\cos(bt), \quad b \in \mathbb{R}$	$\frac{s}{s^2+b^2}$ for $s>0$
$\sin(bt), \quad b \in \mathbb{R}$	$\frac{b}{s^2+b^2}$ for $s>0$
$\frac{t}{2b}\sin(bt), \quad b \in \mathbb{R}\backslash\{0\}$	$\frac{s}{(s^2+b^2)^2}$ for $s>0$
$\frac{1}{2b^3}\left(\sin(bt)-bt\cos(bt)\right), \quad b \in \mathbb{R}\backslash\{0\}$	$\frac{1}{(s^2+b^2)^2}$ for $s>0$

$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a-b)+\sin(a+b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$