Projects in Financial Mathematics

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Preface

The present text contains a number of projects on financial mathematics topics. The students reading this text and carrying out the projects are assumed to be familiar with the fundamental concepts in finance and financial mathematics presented for instance in [2]. A short review of some of these concepts is to be found in Chapter 0 of the present text.

Contents

Chapter 0

Background

0.1 Basic financial concepts

Financial assets

The term **asset** may be used to identify any resource capable of producing value and which, under specific legal terms, can be bought and sold (i.e., converted into cash). Assets may be tangible (e.g., lands, buildings, commodities, etc.) or intangible (e.g., patents, copyrights, stocks, etc.). Assets are also divided into **real assets**, i.e, assets whose value is derived by an intrinsic property (e.g., tangible assets), and **financial assets**, such as stocks, options, bonds, etc., whose value is instead derived from a contractual claim on the income generated by another (possibly real) asset. For example, upon holding shares of the Volvo stock (a financial asset), we can make a profit from the production and sale of cars even if we do not own an auto plant (a real asset). As we consider only financial assets in this text, the terms "asset" and "financial asset" will be henceforth used interchangeably.

Price

The **price** of a financial asset is the value, measured in some units of currency (e.g. dollars), at which the **buyer** and the **seller** agree to exchange ownership of the asset. The price is chosen by the two parties as a result of some kind of "negotiation". More precisely, the **ask price** is the minimum price at which the seller is willing to sell the asset, while the **bid price** is the maximum price that the buyer is willing to pay for the asset. A **transaction** occurs when the bid price of a buyer matches the ask price of a seller, in which case the exchange of the asset takes place at the corresponding price.

A generic financial asset will be denoted by $\mathcal{U}$ and its price at time t by $\Pi^{\mathcal{U}}(t)$. Prices are generally positive, although some financial assets (e.g., forward contracts) have zero price.

The asset price refers to the price per **share** of the asset, where "share" stands for the minimum amount of an asset that can be traded. All prices in this text are given in a fixed currency, which is however left unspecified.

Markets

Financial assets can be traded in **exchange** markets or **over the counter (OTC)**. In the former case all trades are subject to a common regulation, while in the latter the trading conditions are more flexible and, to a certain degree, can be agreed upon by the individual traders. The same asset can be traded both in an exchange market and OTC, usually for a different price. The advantage of trading in regularized exchange markets is the higher level of transparency and protection offered by standardized contracts.

Examples of official exchange markets, respectively of stocks and options, are the Nasdaq market and the Chicago Board of Options Exchange (CBOE); currencies are example of financial assets which are traded only OTC (**Forex market**).

A **market maker** is large investment company that continuously quotes both an ask price and a bid price for immediate purchase/sell of an asset, thereby ensuring markets **liquidity**. The difference between the bid and the ask price of an asset quoted by a market maker is also called the **bid-ask spread** of the asset.

Any transaction in the market is subject to **transaction costs** (e.g., exchange fees) and **transaction delays** (trading in real markets is not instantaneous).

Buyers and sellers of assets in a market will be called **investors** or **agents**.

Long and short position

An investor is said to **short-sell** N shares of an asset if the investor borrows the shares from a third party and then sell them immediately on the market. The reason for short-selling an asset is the expectation that the price of the asset will decrease in the future. In fact, suppose that N shares of an asset $\mathcal{U}$ are short-sold at time $t = 0$ for the price $\Pi^{\mathcal{U}}(0)$ and let $T > 0$. If $\Pi^{\mathcal{U}}(T) < \Pi^{\mathcal{U}}(0)$, then upon re-purchasing the N shares at time T, and returning them to the lender, the short-seller will make the profit $N(\Pi^{\mathcal{U}}(0) - \Pi^{\mathcal{U}}(T))$.

An investor is said to have a **long position** on an asset if the investor profits from an increase of its price (e.g., the investor owns the asset). Conversely, the investor is said to have a **short position** on the asset if the investor will profit from a decrease of its value, as it happens for instance when the investor is short-selling the asset.

Stocks and dividends

The **capital stock** of a company is the part of the company equity capital that is made publicly available for trading. Stocks are most commonly traded in official exchange markets. For instance, over 300 company stocks are traded in the Stockholm exchange market. The price per share of a generic stock at time t will be denoted by $S(t)$.

A stock may occasionally pay a **dividend** to its shareholders, usually in the form of a cash deposit. The amount (per share) of the dividend and its **payment date** must be declared in advance (**announcement date**). The **ex-dividend date** is the first day before the payment date (usually a few days before it) at which buying the stock does not entitle to the dividend. An investor who buys the stock prior to the ex-dividend day and holds it until the ex-dividend day is entitled to the dividend, even if the investor does not own the stock at the payment day. At the ex-dividend day, the price of the stock often (but not always) drops of roughly the same amount paid by the dividend.

Market index and ETF's

A **market index** is a weighted average of the value of a collection of assets traded in one or more exchange markets. For example, S&P500 (Standard and Poor 500) measures the average value of 500 stocks traded at the New York stock exchange (NYSE) and NASDAQ-markets. Market indexes can be regarded themselves as tradable assets. More precisely an **ETF** (Exchange Traded Fund) on a market index is a financial asset whose value tracks exactly the value of the market index (or a given fraction thereof). Hence one share of an ETF on S&P500 will increase its value of 1% in one day if during that day S&P 500 has gained 1%. An **inverse ETF** however will in the same example decrease its value of 1%. Thus ETF's give investors the possibility to speculate whether the market will gain or loose value in the future.

Portfolio position and portfolio process

Consider an agent that invests on N assets $\mathcal{U}_1, \dots, \mathcal{U}_N$ during the time interval $[0, T]$. Assume that the agent trades on a_1 shares of the asset $\mathcal{U}_1$, a_2 shares of the asset $\mathcal{U}_2, \dots$, a_N shares of the asset $\mathcal{U}_N$. Here $a_i \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$, where $a_i < 0$ means that the investor has a short position on the asset $\mathcal{U}_i$, while $a_i > 0$ means that the investor has a long position on the asset $\mathcal{U}_i$ (the reason for this interpretation will become soon clear). The vector $\mathcal{A} = (a_1, a_2, \dots, a_N) \in \mathbb{Z}^N$ is called a **portfolio position**, or simply a portfolio. The **portfolio value** at time t is given by

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a_i \Pi^{\mathcal{U}_i}(t), \quad t \in [0, T], \tag{1}$$

where $\Pi^{\mathcal{U}_i}(t)$ denotes the price of the asset $\mathcal{U}_i$ at time t. The value of the portfolio measures the wealth of the investor: the higher is $V(t)$, the "richer" is the investor at time t. It follows that when the price of the asset $\mathcal{U}_i$ increases, the value of the portfolio increases if $a_i > 0$ and decreases if $a_i < 0$, hence, as stated above, $a_i > 0$ corresponds to a long position on the asset $\mathcal{U}_i$ and $a_i < 0$ to a short position. Portfolios can be added by using the linear structure on $\mathbb{Z}^N$, namely if $\mathcal{A}, \mathcal{B} \in \mathbb{Z}^N$, $\mathcal{A} = (a_1, \dots, a_N)$, $\mathcal{B} = (b_1, \dots, b_N)$ are two portfolios and $\alpha, \beta \in \mathbb{Z}$, then $\mathcal{C} = \alpha\mathcal{A} + \beta\mathcal{B}$ is the portfolio $\mathcal{C} = (\alpha a_1 + \beta b_1, \dots, \alpha a_N + \beta b_N)$, whose value is given by $V_{\mathcal{C}}(t) = \alpha V_{\mathcal{A}}(t) + \beta V_{\mathcal{B}}(t)$.

In the definition of portfolio position and portfolio value given above, the investor keeps the same number of shares of each asset during the whole time interval $[0, T]$. Suppose now that the investor changes the position on the assets at some times $t_1, \dots, t_{M-1}$, where

$$0 = t_0 < t_1 < t_2 < \dots < t_{M-1} < t_M = T;$$

for simplicity we assume that at each time $t_1, \dots, t_{M-1}$ the change in the portfolio position occurs instantaneously. Let $\mathcal{A}_0$ denote the initial (at time $t = t_0 = 0$) portfolio position of the investor and $\mathcal{A}_j$ denote the portfolio position of the investor in the interval of time $(t_{j-1}, t_j]$, $j = 1, \dots, M$. As positions hold for one instance of time only are clearly meaningless, we may assume that $\mathcal{A}_0 = \mathcal{A}_1$, i.e., $\mathcal{A}_1$ is the portfolio position in the closed interval $[0, t_1]$. The vector $(\mathcal{A}_1, \dots, \mathcal{A}_M)$ is called a **portfolio process**. Denoting by a_{ij} the number of shares of the asset i in the portfolio $\mathcal{A}_j$, a portfolio process is equivalent to the $N \times M$ matrix $A = (a_{ij})$, $i = 1, \dots, N$, $j = 1, \dots, M$. The value $V(t)$ of the portfolio process at time t is given by the value of the corresponding portfolio position at time t as defined by (1), that is

$$V(t) = \begin{cases} V_{\mathcal{A}_1}(t) = \sum_{i=1}^N a_{i1} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in [0, t_1] \\ V_{\mathcal{A}_2}(t) = \sum_{i=1}^N a_{i2} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in (t_1, t_2] \\ \vdots & \vdots \\ V_{\mathcal{A}_M}(t) = \sum_{i=1}^N a_{iM} \Pi^{\mathcal{U}_i}(t), & \text{for } t \in (t_{M-1}, t_M] \end{cases} .$$

The initial value $V(0) = V_{\mathcal{A}_0} = V_{\mathcal{A}_1}(0)$ of the portfolio, when it is positive, is called the **initial wealth** (or **capital**) of the investor.

A portfolio process is said to be **self-financing** if the portfolio assets pay no dividends and if no cash is ever withdrawn or infused in the portfolio. For example, let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ be non-dividend paying assets in the interval $[0, T]$. Suppose that at time $t_0 = 0$ the investor is short 400 shares on the asset $\mathcal{U}_1$, long 200 shares on the asset $\mathcal{U}_2$ and long 100 shares on the asset $\mathcal{U}_3$. This corresponds to the portfolio

$$\mathcal{A}_0 = (-400, 200, 100),$$

whose value is

$$V_{\mathcal{A}_0} = -400\, \Pi^{\mathcal{U}_1}(0) + 200\, \Pi^{\mathcal{U}_2}(0) + 100\, \Pi^{\mathcal{U}_3}(0).$$

If this value is positive, the investor needs an initial wealth to set up this portfolio position: the income derived from short selling the asset $\mathcal{U}_1$ does not suffice to open the desired long position on the other two assets. As mentioned before, we may assume that the investor keeps the same position in the interval $(0, t_1]$, i.e., $\mathcal{A}_1 = \mathcal{A}_0$. The value of the portfolio process at time $t = t_1$ is

$$V(t_1) = V_{\mathcal{A}_1}(t_1) = -400\,\Pi^{\mathcal{U}_1}(t_1) + 200\,\Pi^{\mathcal{U}_2}(t_1) + 100\,\Pi^{\mathcal{U}_3}(t_1).$$

Now suppose that at time $t = t_1$ the investor buys 500 shares of $\mathcal{U}_1$, sells x shares of $\mathcal{U}_2$, and sells all the shares of $\mathcal{U}_3$. Then in the interval $(t_1, t_2]$ the investor has the new portfolio position given by

$$\mathcal{A}_2 = (100, 200 - x, 0),$$

and so the value of the portfolio process for $t \in (t_1, t_2]$ is given by

$$V(t) = 100\,\Pi^{\mathcal{U}_1}(t) + (200 - x)\,\Pi^{\mathcal{U}_2}(t), \quad t \in (t_1, t_2].$$

The limit of this quantity as $t \to t_1^+$ corresponds to the value of the portfolio "immediately after" the position has been changed at time t_1. Denoting

$$V(t_1^+) = \lim_{t \to t_1^+} V(t)$$

and assuming that the prices are continuous, we have

$$V(t_1^+) = 100\,\Pi^{\mathcal{U}_1}(t_1) + (200 - x)\,\Pi^{\mathcal{U}_2}(t_1).$$

The difference between the value of the two portfolios immediately after and immediately before the transaction is then

$$\begin{aligned} V(t_1^+) - V(t_1) &= 100\,\Pi^{\mathcal{U}_1}(t_1) + (200 - x)\,\Pi^{\mathcal{U}_2}(t_1) \\ &\quad - (-400\,\Pi^{\mathcal{U}_1}(t_1) + 200\,\Pi^{\mathcal{U}_2}(t_1) + 100\,\Pi^{\mathcal{U}_3}(t_1)) \\ &= 500\,\Pi^{\mathcal{U}_1}(t_1) - x\,\Pi^{\mathcal{U}_2}(t_1) - 100\,\Pi^{\mathcal{U}_3}(t_1). \end{aligned}$$

If this difference is positive, then the new portfolio cannot be created from the old one without infusing extra cash. Conversely, if this difference is negative, then the new portfolio is less valuable than the old one, the difference being equivalent to cash withdrawn from the portfolio. Hence for self-financing portfolio processes we must have $V(t_1^+) - V(t_1) = 0$, and similarly $V(t_j^+) - V(t_j) = 0$, for all $j = 1, \dots, M - 1$. This implies in particular that the number x of shares of the asset $\mathcal{U}_2$ to be sold at time t_1 in a self-financing portfolio must be

$$x = \frac{500\Pi^{\mathcal{U}_1}(t_1) - 100\Pi^{\mathcal{U}_3}(t_1)}{\Pi^{\mathcal{U}_2}(t_1)}.$$

Of course, x will be an integer only in exceptional cases, which means that perfect self-financing strategies in real markets are almost impossible.

If $V(t_j^+) \neq V(t_j)$, i.e., if the portfolio value is discontinuous at time t_j, we say that the portfolio process generates the **cash flow**

$$C(t_j) = -(V(t_j^+) - V(t_j))$$

at time t_j. A positive cash flow corresponds to cash *removed* from the portfolio (causing a decrease of its value), while a negative cash flow corresponds to cash *added* to the portfolio. For instance if at time t_1 the investor sells shares of $\mathcal{U}_1$ and the income is not used to buy shares of another asset, i.e., if it is removed from the portfolio, then $V(t_1^+) < V(t_1)$ and thus $C(t_1) > 0$. The total cash flow generated by the portfolio process in the interval $[0, T]$ is $C_{\mathrm{tot}} = \sum_{j=1}^{M-1} C(t_j)$ and can be negative, positive or zero.

If $C(t_1) = \cdots = C(t_{M-1}) = 0$ and the assets pay no dividends, the portfolio process is self-financing.

If an asset pays a dividend D at some time $t_* \in (0, T)$, then the portfolio process generates the positive cash flow D at time t_* if the portfolio is long on the asset and the negative cash flow $-D$ if it is short on the asset (because the dividend is due to the original owner of the asset). Constant portfolio positions are self-financing provided the assets pay no dividends.

Portfolios and assets return

Suppose that a portfolio process is opened at time $t = 0$ and closed at time $t = T > 0$, i.e., all positions in the portfolio are liquidated at time T. If the portfolio process is self-financing, then its **return** in the interval $[0, T]$ is given by

$$R(T) = V(T) - V(0), \tag{2}$$

where $V(t)$ denotes the value of the portfolio at time t. The quantity $V(T)$ is also called the **pay-off** of the portfolio. If the portfolio return is positive, the investor makes a **profit** in the interval $[0, T]$, if it is negative the investor incurs in a **loss**. When $V(0) > 0$ we may also compute the **rate of return** of the portfolio, which is given by

$$R_{\mathrm{rate}}(T) = \frac{V(T) - V(0)}{V(0)} \quad \text{(expressed in \%)}. \tag{3}$$

The total cash flow C generated by a (non-self-financing) portfolio process must be included in the computation of the return of the portfolio in the interval $[0, T]$ according to the formula

$$R(T) = V(T) - V(0) + C. \tag{4}$$

Portfolio returns are commonly **annualized** by dividing the return $R(T)$ by the time T expressed in fraction of years (e.g., $T = 6$ months $= 1/2$ years).

Consider now a portfolio that consists of a long position on one share of the asset $\mathcal{U}$ in the interval $[t, t + h]$ and assume that the asset pays no dividend in this time interval. The annualized rate of return of this portfolio is given by

$$R_h(t) = \frac{\Pi^{\mathcal{U}}(t + h) - \Pi^{\mathcal{U}}(t)}{h\, \Pi^{\mathcal{U}}(t)}$$

and is also called **simply compounded rate of return** of $\mathcal{U}$. In the limit $h \to 0^+$ we obtain the **continuously compounded** (or **instantaneous**) **rate of return** of the asset:

$$r(t) = \lim_{h\to 0^+} R_h(t) = \frac{1}{\Pi^{\mathcal{U}}(t)}\frac{d\Pi^{\mathcal{U}}(t)}{dt},$$

where we assume that the price of $\mathcal{U}$ is differentiable in time.

Asset returns are often computed using the logarithm of the price rather than the price itself. For instance the quantity[1]

$$\widehat{R}_h(t) = \log \Pi^{\mathcal{U}}(t+h) - \log \Pi^{\mathcal{U}}(t) = \log\left(\frac{\Pi^{\mathcal{U}}(t+h)}{\Pi^{\mathcal{U}}(t)}\right)$$

is called **simply compounded log-return** of the asset $\mathcal{U}$ in the interval $[t, t+h]$. The use of the log-price is convenient in some computations because $\Pi^{\mathcal{U}}(t) > 0$, while $\log \Pi^{\mathcal{U}}(t) \in \mathbb{R}$, i.e., the boundary at zero of the asset price is removed when the log-price is employed. Since $\widehat{R}_h(t)/h$ and $R_h(t)$ have the same limit when $h \to 0^+$, namely

$$\lim_{h\to 0^+}\frac{1}{h}\widehat{R}_h(t) = \lim_{h\to 0^+}\frac{\log \Pi^{\mathcal{U}}(t+h) - \log \Pi^{\mathcal{U}}(t)}{h} = \frac{d\log \Pi^{\mathcal{U}}(t)}{dt} = r(t),$$

then $r(t)$ is also called **continuously compounded** (or **instantaneous**) **log-return** of the asset. Note carefully that in general $\widehat{R}_h(t)$, $R_h(t)$ and $r(t)$ are *not* known at time t, because they depend on the future value of the asset $\mathcal{U}$; an exception to this are money market assets discussed later.

Historical volatility

The historical volatility of an asset measures the amplitude of the time fluctuations of the asset price, thereby giving information on its level of uncertainty. It is computed as the standard deviation of the log-returns of the asset based on historical data. More precisely, let $[t_0, t]$ be some interval of time in the past, with t denoting possibly the present time, and let $T = t - t_0 > 0$ be the length of this interval. Let us divide $[t_0, t]$ into n equally long periods, say

$$t_0 < t_1 < t_2 < \cdots < t_n = t, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \dots, n.$$

The set of points $\{t_0, t_1, \dots, t_n\}$ is called a **uniform partition** of the interval $[t_0, t]$. Assume for instance that the asset is a stock. Denote the log-return of the stock price in the interval $[t_{i-1}, t_i]$ as

$$\widehat{R}_i = \log S(t_i) - \log S(t_{i-1}) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right), \quad i = 1, \dots, n. \tag{5}$$

[1]Throughout this text, $\log x$ stands for the natural logarithm of $x > 0$, which is also frequently denoted by $\ln x$ in the literature.

The average of the log-returns is

$$\widehat{R}(t) = \frac{1}{n}\sum_{i=1}^{n}\widehat{R}_i = \frac{1}{n}\log\left(\frac{S(t)}{S(t_0)}\right). \tag{6}$$

The **T-historical mean of log-return** of the stock is obtained by annualizing the average of log-returns, i.e., by dividing $\widehat{R}(t)$ by the length h of the interval in which the log returns are computed:

$$\alpha_T(t) = \frac{1}{nh}\log\left(\frac{S(t)}{S(t_0)}\right) = \frac{1}{T}\log\left(\frac{S(t)}{S(t_0)}\right) \quad (T\text{-historical mean of log-return}). \tag{7}$$

The (corrected) sample variance of the log-returns is

$$\Delta(t) = \frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2.$$

The **T-historical variance** of the stock is obtained by annualizing $\Delta(t)$, i.e.,

$$\sigma_T^2(t) = \frac{1}{h}\frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2 \quad (T\text{-historical variance}). \tag{8}$$

The square root of the $T-$historical variance is the **T-historical volatility** of the stock:

$$\sigma_T(t) = \frac{1}{\sqrt{h}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2} \quad (T\text{-historical volatility}). \tag{9}$$

Note carefully that, as opposed to the historical mean, the historical volatility of the stock depends not only on the stock prices in the time interval $[t_0, t]$ but also on the chosen partition of this interval.

Suppose for example that $t - t_0 = T = 20$ days, which is quite common in the applications, and let $t_1, \dots, t_{20}$ be the market closing times at these days. Let $h = 1$ day $= 1/365$ years. Then

$$\sigma_{20}(t) = \sqrt{365}\sqrt{\frac{1}{19}\sum_{i=1}^{n}(\widehat{R}_i - \widehat{R}(t))^2}$$

is called the 20days-historical volatility. Two examples of the curve $t \to \sigma_{20}(t)$ are shown in Figure 1.

Remark 0.1. The factor $h = 1/252$ is also commonly used in the calculation of market parameters, since there are 252 trading days in one year (markets are closed in the week-end).

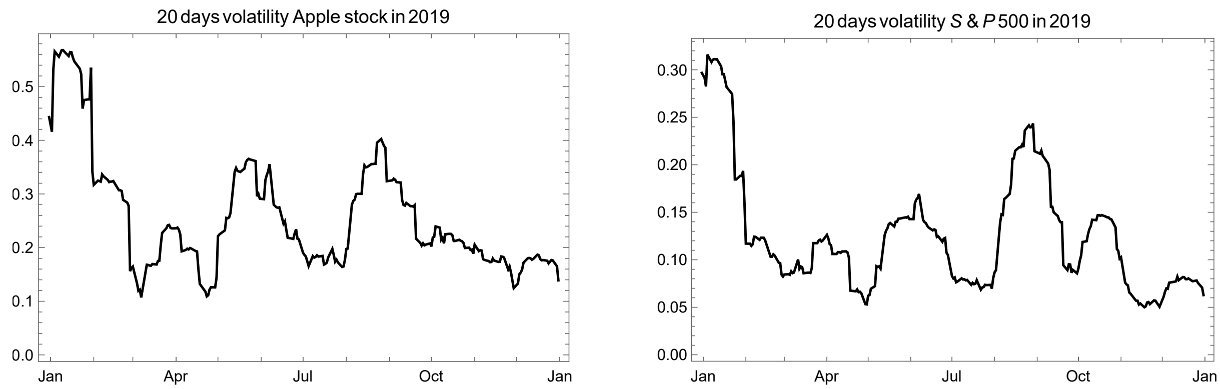


Figure 1: 20-days volatility of the Apple stock and the S&P500 index from January 1st, 2019 until December 31st, 2019.

Assets correlation

Consider again a uniform partition $\{t_0, \dots, t_n = t\}$ of the past interval $[t_0, t]$ with length $T = t - t_0$. Let $S^{(1)}(t)$, $S^{(2)}(t)$ be the prices of two stocks. Let $\widehat{R}_i^{(1)}, \widehat{R}_i^{(2)}$ be the log-returns of each stock in the interval $[t_{i-1}, t_i]$ and $R^{(1)}, R^{(2)}$ be the averages of log-returns. The **T-historical correlation of log-returns** is computed with the formula

$$\rho_T(t) = \frac{\sum_{i=1}^n (\widehat{R}_i^{(1)} - \widehat{R}^{(1)})(\widehat{R}_i^{(2)} - \widehat{R}^{(2)})}{\sqrt{\sum_{i=1}^n (\widehat{R}_i^{(1)} - \widehat{R}^{(1)})^2 \sum_{i=1}^n (\widehat{R}_i^{(2)} - \widehat{R}^{(2)})^2}}. \tag{10}$$

Denoting by a_1, a_2 the n-dimensional vectors $a_j = (\widehat{R}_1^{(j)} - \widehat{R}^{(j)}, \widehat{R}_2^{(j)} - \widehat{R}^{(j)}, \dots, \widehat{R}_n^{(j)} - \widehat{R}^{(j)})$, $j = 1, 2$, we can rewrite $\rho_T(t)$ as

$$\rho_T(t) = \frac{a_1 \cdot a_2}{|a_1||a_2|} = \cos\theta,$$

where $\cdot$ denotes the inner product of vectors, $|a_j|$ is the norm of the vector a_j and $\theta \in [0, \pi]$ is the angle between a_1 and a_2. Hence $\rho_T(t) \in [-1, 1]$ and the closer is $\rho_T(t)$ to 1 (resp. -1) the more the stock prices have tendency to move in the same (resp. opposite) direction. For instance, it is clear from Figure 1 that the Apple stock and S&P 500 were strongly positively correlated during the period of time reported in the figure.

Financial derivatives. Options

A **financial derivative** (or **derivative security**) is an asset whose value depends on the performance of one (or more) other asset(s), which is called the **underlying asset**. There exist several types of financial derivatives, the most common being options, futures, forwards and swaps. Derivatives are available on many different types of underlying assets, including currencies, market indexes, bonds, commodities, etc. In this section we discuss option derivatives on a single asset, which could be for instance a stock.

A **call option** is a contract between two parties: the **buyer**, or **owner**, of the call and the **seller**, or **writer**, of the call. The contract gives the owner the right, but *not* the obligation, to buy the underlying asset in the future for a price fixed at the time when the contract is stipulated, and which is called **strike price** of the call. If the buyer can exercise this right only at some given time T in the future then the call option is called **European**, while if the option can be exercised at any time earlier than or equal to T, then the option is called **American**. The time T is called **maturity time**, or **expiration date** of the call. The writer of the call is obliged to sell the asset to the buyer if the latter decides to exercise the option. If the option to buy in the definition of a call is replaced by the option to sell, then the option is called a **put option**.

In exchange for the option, the buyer must pay a **premium** to the seller (options are not free). Suppose that the option is a European option with strike price K and maturity T. Assume that the underlying asset is a stock with price $S(t)$ at time $t \leq T$ and let Π_0 be the premium paid by the buyer to the seller. In which case is it then convenient for the buyer to exercise the option at maturity? Let us define the **pay-off** of the European call as

$$Y_{\text{call}} = (S(T) - K)_+ := \max(0, S(T) - K) = \begin{cases} 0 & \text{if } S(T) \leq K \\ S(T) - K & \text{if } S(T) > K \end{cases} .$$

Similarly, the pay-off of the European put is defined by

$$Y_{\text{put}} = (K - S(T))_+ = \begin{cases} 0 & \text{if } S(T) \geq K \\ K - S(T) & \text{if } S(T) < K \end{cases} .$$

Clearly, the buyer should exercise the call option at maturity if and only if $Y_{\text{call}} > 0$, as in this case it is cheaper to buy the stock at the strike price rather than at the market price. Similarly the owner of the put should exercise if and only if $Y_{\text{put}} > 0$, as in this case the income generated by selling the stock at the strike price is higher then the income generated by selling it at the market price. Hence the call or put option must be exercised at maturity if and only if the pay-off is positive, in which case the option is said to **expire in the money**. The return for the owner of the option is given by $N(Y_{\text{call}} - \Pi_0)$ in the case of the call and by $N(Y_{\text{put}} - \Pi_0)$ in the case of the put, where N is the number of option contracts in the buyer portfolio. Note carefully that the buyer makes a profit only if the pay-off is greater than the premium. One of the main problems in options pricing theory is to define a reasonable fair value for the price Π_0 of options (and other derivatives).

Let us introduce some further terminology. The European call (resp. put) with strike K is said to be **in the money** at time t if $S(t) > K$ (resp. $S(t) < K$). The call (resp. put) is said to be **out of the money** at time t if $S(t) < K$ (resp. $S(t) > K$). If $S(t) = K$, the (call or put) option is said to be **at the money** at time t. The meaning of this terminology is self-explanatory, see Figure 2.

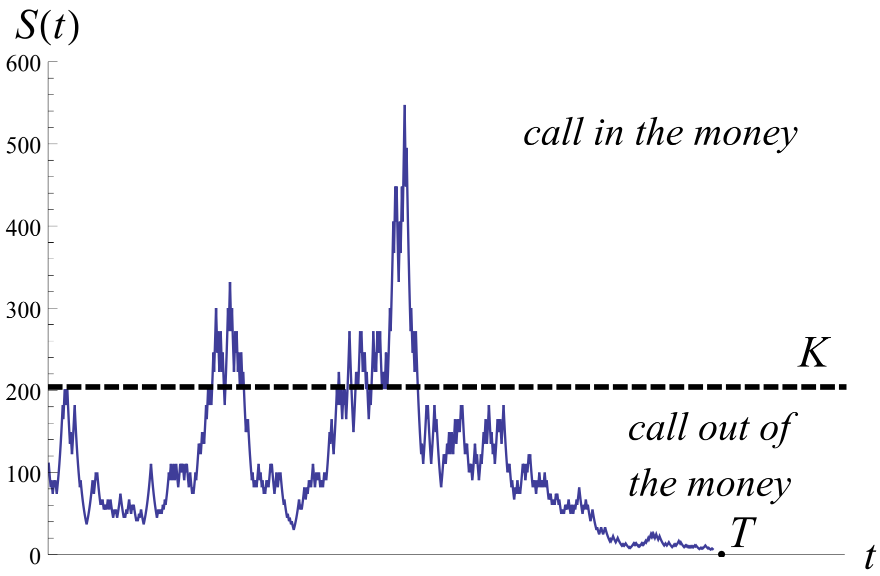


Figure 2: The call option with strike $K = 9.5$ is in the money in the upper region and out of the money in the lower region. The put option with the same strike is in the money in the lower region and out of the money in the upper region.

The pay-off of the American call exercised at time t is $Y(t) = (S(t) - K)_+$, while for the American put it is given by $Y(t) = (K - S(t))_+$. The quantity $Y(t)$ is also called **intrinsic value** of the American option. In particular, the intrinsic value of an out-of-the-money American option is zero.

Option markets

Option markets are relatively new compared to stock markets. The first one has been established in Chicago in 1974 (the Chicago Board Options Exchange, **CBOE**). Market options are available on different assets (stocks, debts, indexes, etc.) and with different strikes and maturities. Most commonly, market options are of American style.

Clearly, the deeper in the money is the option, the higher will be its price in the market, while the price of an option deeply out of the money is usually quite low (but still positive). It is also clear that the buyer of the option is the party holding the long position on the option, since the buyer owns the option and thus hopes for an increase of its value, while the writer is the holder of the short position.

One reason why investors buy call options is to protect a short position on the underlying asset. Suppose for instance that an investor short-sells 100 shares of a stock at time $t = 0$ for the price $S(0)$, expecting that the price of the stock will decrease in the future. At the same time, to alleviate the risk derived from the stock price moving in the opposite direction, the investor buys 100 shares of the American call option on the stock with strike $K \approx S(0)$ and maturity $T > 0$. If at some time $t_0 \in (0, T)$ the price of the stock is no lower than $S(0)$, the investor has the option to exercise the call, obtain 100 shares of the stock for the price $K \approx S(0)$ and thus close the short position on the stock with reduced losses. At the same fashion, investors buy put options to protect a long position on the underlying asset. A trading position (particularly a short position) that is not covered by a suitable security is said to be **naked**.

Of course, **speculation** is also an important factor in option markets. However *the standard theory of options pricing is firmly based on the interpretation of options as derivative securities and does not take speculation into account.*

European, American and Asian derivatives

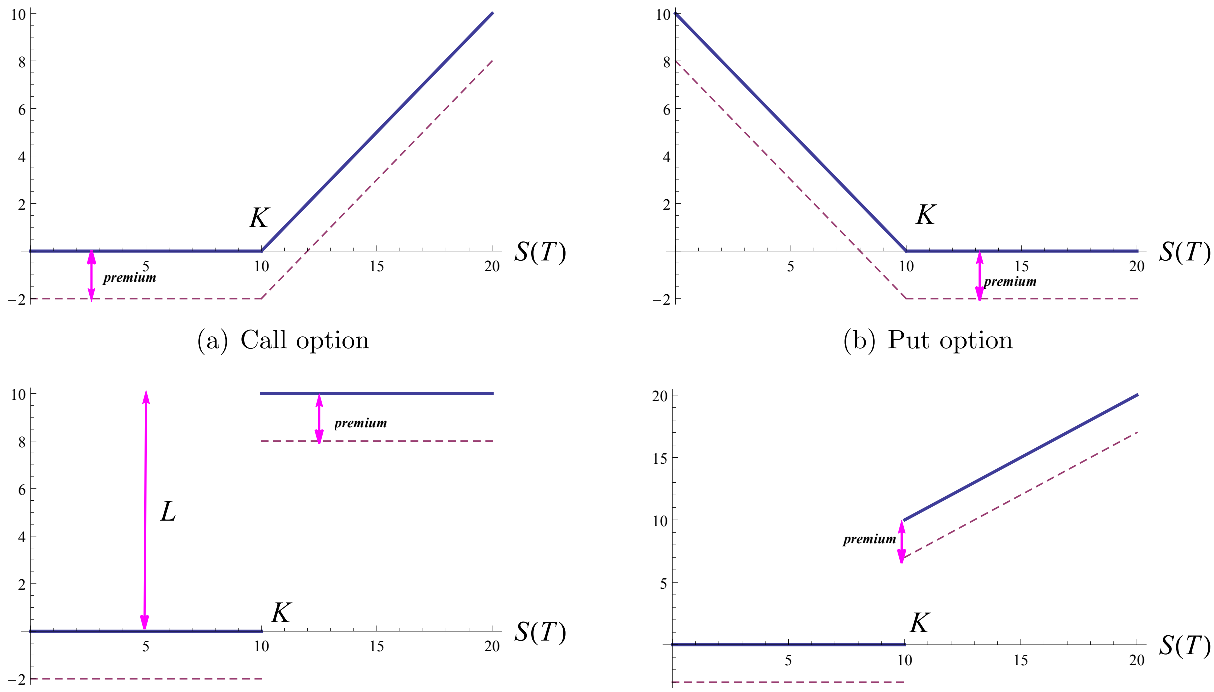
By far the majority of financial derivatives, including options other than simple calls and puts, are traded OTC. Before discussing a few examples, it is convenient to introduce a precise mathematical definition of European and American derivatives.

Given a function $g : (0, \infty) \to \mathbb{R}$, the **standard European derivative** with pay-off $Y = g(S(T))$ and maturity time $T > 0$ is the contract that pays to its owner the amount Y at time $T > 0$. Here $S(T)$ is the price of the underlying stock at time T, while g is the **pay-off function** of the derivative (e.g., $g(x) = (x - K)_+$ for European call options, while $g(x) = (K - x)_+$ for European put options). Hence, the pay-off of standard European derivatives depends only on the price of the stock at maturity and not on the earlier history of the stock price. An important example of standard European derivative (other than call and put options) is the **digital option**. Denote by $H(x)$ the **Heaviside function**,

$$H(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}, \tag{11}$$

and let $K, L > 0$ be constants expressed in units of some currency (e.g., dollars). The standard European derivative with pay-off function $g(x) = LH(x-K)$ is called **cash settled digital call option** with strike price K and **notional value** L; this derivative pays the amount L if $S(T) > K$, and nothing otherwise. The **physically settled digital call option** has the pay-off function $g(x) = xH(x - K)$, which means that at maturity the buyer receives either the stock (when $S(T) > K$), or nothing. Digital options are also called **binary** options. Figure 3 shows the graph of the pay-off function for call, put and digital call options with strike $K = 10$. Drawing the graph of the pay-off function of a derivative helps to get a first insight on its properties.

If the pay-off depends on the history of the stock price during the interval $[0, T]$, and not just on $S(T)$, the contract will be called **non-standard** European derivative. An example of non-standard European derivative is the so-called **Asian call option**, the pay-off of which



(a) Call option

(b) Put option

(c) Digital call option (cash-settled)

(d) Digital call option (physically-settled)

Figure 3: Pay-off function (continuous line) and return (dashed line) of some standard European derivatives.

is given by

$$Y = \left(\frac{1}{T} \int_0^T S(t)\, dt - K \right)_+ .$$

The price at time t of the European derivative with pay-off Y and maturity T will be denoted by $\Pi_Y(t)$ (the expiration date is not included in the notation).

The term "European" signifies that the contract cannot be exercised before maturity T. For a **standard American derivative** the buyer can exercise the contract at any time $t \in (0, T]$ and so doing the buyer will receive the amount $Y(t) = g(S(t))$, where g is the pay-off function of the American derivative. Non-standard American derivatives can be defined similarly to the European ones, but with the further option of earlier exercise. The price at time t of the American derivative with intrinsic value $Y(t)$ and maturity T will be denoted $\widehat{\Pi}_Y(t)$.

Remark 0.2. The terminology "standard" and "non-standard" derivative is used in this text for easy reference. It is *not* employed in the financial world.

Forward contracts

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a European type financial derivative stipulated by two parties in which one agrees to sell (and possibly deliver) to the other the asset $\mathcal{U}$ at time T in exchange for the cash K. As opposed to options, forward contracts give the same right/obligation to the two parties, as they are both *obliged* to fulfil their part of the agreement at maturity T (buy or sell the asset for the price K). In particular, as there is no privileged position in a forward contract, neither of the two parties has to pay a premium when the contract is stipulated, that is to say, *forward contracts are free*; in fact, the terminology used for forward contracts is "to enter a forward contract" and not "to buy/sell a forward contract". The party who must sell the asset at maturity is said to hold the short position on the forward, while the party who must buy the asset is said to hold the long position, although strictly speaking this terminology refers to the type of position on the underlying asset rather than on the forward contract (which has zero value at all times). Hence the pay-off for a long position in a forward contract on the asset $\mathcal{U}$ is

$$Y_{\text{long}} = (\Pi^{\mathcal{U}}(T) - K),$$

while for the holder of the short position the pay-off is

$$Y_{\text{short}} = (K - \Pi^{\mathcal{U}}(T)).$$

Forward contracts are traded OTC and most commonly on commodities or market indexes, such as currency exchange rates, interest rates and volatilities. In the case that the underlying asset is an index, forward contracts are also called **swaps** (e.g., currency swaps, interest rate swaps, volatility swaps, etc.).

One purpose of forward contracts is to share risks. Irrespective of the movement of the underlying asset in the market, its price at time T for the holders of the forward contract will be K. The delivery price agreed by the two parties in a forward contract is also called the **forward price** of the asset. More precisely, the T-forward price $\mathrm{For}_{\mathcal{U}}(t,T)$ of an asset $\mathcal{U}$ at time $t < T$ is the delivery price of a forward contract on $\mathcal{U}$ stipulated at time t and with maturity T, while the current, actual price $\Pi^{\mathcal{U}}(t)$ of the asset is called the **spot price**.

As forward contracts are traded OTC, the forward price of an asset is not an objective parameter (as opposed to the future price defined below), since different investors can agree on different delivery prices at the same maturity date.

Futures contracts

Futures are standardized forward contracts listed in official exchange markets, called **futures market**, which include for instance the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the

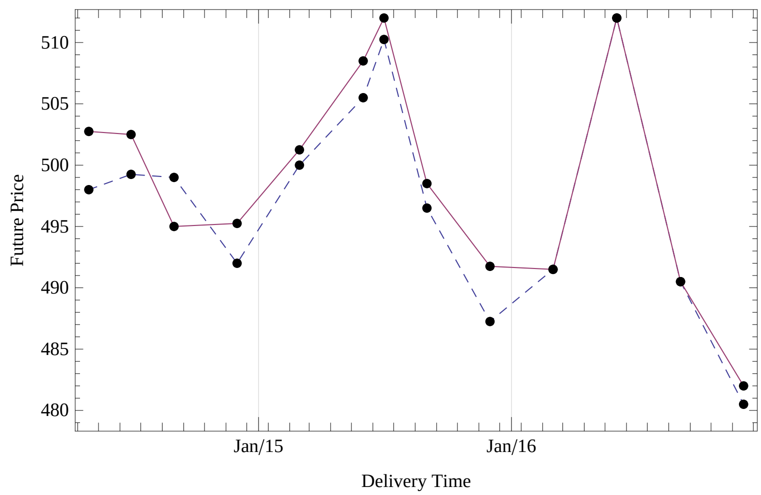


Figure 4: Futures price of corn on May 12, 2014 (dashed line) and on May 13, 2014 (continuous line) for different delivery times

International Exchange Group (ICE). Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation. In particular, at any given time the price for delivery of an asset at a fixed time in the future is the same for all investors. The T-**future price** $\mathrm{Fut}_{\mathcal{U}}(t,T)$ of the asset $\mathcal{U}$ at time $t \leq T$ is defined as the delivery price at time $t \leq T$ in the futures contract with maturity T on the asset $\mathcal{U}$.

Holding a position in a futures contract in the futures market consists in the agreement to receive as a cash flow the change in the future price of the underlying asset during the time in which the position is held. The cash flow may be positive or negative. In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down. The cash flow is distributed in time through the so called **margin account**. For example, assume that at $t = 0$ an investor opens a long position in a futures contract expiring at time T. At the same time, the investor needs to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened). At $t = 1$ day, the amount $\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}(0,T)$ will be added to the account, if it positive, or withdrawn, if it is negative. The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$(\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(t-1,T)) + (\mathrm{Fut}_{\mathcal{U}}(t-1,T) - \mathrm{Fut}_{\mathcal{U}}(t-2,T)) + \\ \cdots + (\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)) = (\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)).$$

If the long position is held up to the time of maturity, then the investor should buy the underlying asset. However futures contracts are often **cash settled** and not **physically settled**, which means that the delivery of the underlying asset does not occur, and the equivalent value in cash is paid instead.

An **option on futures** with maturity $T > 0$ and strike K is a contract that gives to the owner the right to enter at time T in a futures contract (expiring at time $S > T$) at the future price K. In the case of a call (resp. put) option, the owner has the right to take a long (resp. short) position on the futures contract and thus the pay-off will be $(\mathrm{Fut}_{\mathcal{U}}(T, S) - K)_+$ (resp. $(K - \mathrm{Fut}_{\mathcal{U}}(T, S))_+$). If the option on futures expires in the money, the owner can decide to keep open the position on the futures contract or to close it immediately, thereby cashing the pay-off of the option. Options on futures are example of **second derivatives**, i.e., financial derivatives whose underlying asset is another derivative.

Bonds

The **zero coupon bond** (**ZCB**) with **face** (or **nominal**) **value** K and maturity $T > 0$ is the contract that promises to pay to its owner the amount K at time T in the future. Without loss of generality it will be assumed from now on that $K = 1$, as owning one share of the ZCB with face value K is clearly equivalent to own K shares of the ZCB with face value 1. ZCB's (and the related coupon bonds described below) are first issued in the so-called **primary market** by national governments and private companies as a way to borrow money and fund their activities; starting from the following market day, the ZCB's become tradable assets in the **secondary market** and thus their price changes in time. Let $B(t, T)$ denote the price at time t of the ZCB with face value 1 and expiring at time T. If the issuer of the ZCB announces at time $t_0 < T$ that it is unable to comply with the payment of the face value at maturity, then the ZCB becomes worthless, i.e., $B(t, T) \equiv 0$ for $t \in [t_0, T]$ and the issuer of the ZCB is said to be in **default**. Suppose that the issuer of the ZCB bears no risk of default in the interval $[t, T]$. The investors who own shares of the ZCB at maturity T will then receive at time T the promised face value, multiplied by the number of shares owned, from the original issuer of the ZCB. The return per share of this investment is $R(t) = 1 - B(t, T)$, where t is the time at which the investor bought the ZCB. Under normal market conditions, $B(t, T) < 1$, for $t < T$, i.e., the investor pays less than 1 today to receive 1 in the future, and thus $R(t) > 0$. However exceptions are possible; for instance national bonds in Sweden with maturity shorter than 10 years yield currently (2020) a negative return.

Bonds with long maturity typically pay coupons in addition to the face value. Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0, T]$. A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \ldots, c_M \in (0, 1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1 + c_M$ at maturity $T = t_M$. Most commonly the coupons are all equal, i.e., $c_1 = c_2 = \cdots = c_M$, and paid annually (or semi-annually). The maturity of coupon bonds can reach up to 30 or more years.

Money market

The **money market** is a component of the debt market consisting of **short term loans**, i.e., loan contracts with maturity between one day and one year. Examples of money market assets are **treasury bills**, i.e., ZCB's with short maturity (less than 1 year), commercial papers, certificates of deposit, saving accounts and repurchase agreements (**repo**). In contrast to stock and option markets, money markets are typically accessible only by financial institutions and not by private investors.

The value at time t of a generic asset in the money market will be denoted by $B(t)$. The difference $B(t_2) - B(t_1)$, $t_1 < t_2$, determines the **interest rate** of the asset in the interval $[t_1, t_2]$. In particular, let $\{t_0 = 0, t_1, \dots, t_N = t\}$ be a uniform partition of the interval $[0, t]$ with size $h = t_i - t_{i-1}$. The money market asset is said to have **simply compounded** interest rate $R_h(s)$ in the time period $[s, s+h]$, where $s \in \{t_0, \dots, t_{N-1}\}$, if the value of the asset satisfies

$$B(s+h) = B(s)(1 + R_h(s)h), \quad s \in \{t_0, \dots, t_{N-1}\}. \tag{12}$$

Inverting (12) we have

$$R_h(s) = \frac{B(s+h) - B(s)}{hB(s)}, \tag{13}$$

i.e., $R_h(s)$ is the annualized rate of return of the asset in the interval $[s, s+h]$. Note carefully that $R_h(s)$ *is known at time* s (as opposed for instance to the return of stocks in the interval $[s, s+h]$, which is not known at time s). Iterating (12) the value at time $t = t_N$ of the risk-free asset can be expressed in terms of the value at time $t = 0$ by the formula

$$\begin{aligned} B(t) &= B(t_{N-1})(1 + R_h(t_{N-1})h) = B(t_{N-2})(1 + R_h(t_{N-2})h)(1 + R_h(t_{N-1})h) \\ &= \cdots = B(0) \prod_{i=0}^{N-1} (1 + R_h(t_i)h). \end{aligned} \tag{14}$$

Example. Suppose that at time $t_0 = 0$ an investor is borrowing the quantity $B(0) =$ 1000000 Kr for one year with 3-months compounded interest rate, i.e., $h = 1/4$. Suppose $R_{1/4}(t_0) = 0.03$ in the first quarter, $R_{1/4}(t_1) = 0.02$ in the second quarter, $R_{1/4}(t_2) = 0.01$ in the third quarter and $R_{1/4}(t_3) = 0.04$ in the last quarter. Here $t_0 = 0$, $t_1 = 1/4$, $t_2 = 1/2$, $t_3 = 3/4$. The debt of the investor at time $t_4 = 1$ year is

$$B(t_4) = B(t_0)(1 + \frac{1}{4}R_{1/4}(t_0))(1 + \frac{1}{4}R_{1/4}(t_1))(1 + \frac{1}{4}R_{1/4}(t_2))(1 + \frac{1}{4}R_{1/4}(t_3)) \approx 1025220 \text{ Kr}.$$

If the investor borrows instead at the yearly compounded rate $R_1(t_0) = 0.03$ (i.e., $h = 1$), the debt after 1 year is $B(t_4) = B(t_0)(1 + R_1(t_0)) = 1030000$ Kr. Notice that at time $t = t_0$ the investor knows $R_{1/4}(t_0)$ and $R_1(t_0)$ but does not know the values of $R_{1/4}(t_1), R_{1/4}(t_2), R_{1/4}(t_3)$ and thus cannot anticipate whether it is more convenient to borrow at variable or constant interest rate. Investors may use financial instruments such as **interest rate swaps** or

interest rate caps/floors to hedge against the risk derived from the fluctuations of interest rates in the market.

Letting $h \to 0$ in (13) we obtain the **continuously compounded** interest rate (or **short rate**) $r(t)$ of the money market asset, namely

$$R_h(s) \to r(s) = \frac{B'(s)}{B(s)} = \frac{d}{ds} \log B(s), \quad \text{as } h \to 0. \tag{15}$$

Thus $r(t)$ is the interest rate to borrow at time t for an "infinitesimal" interval of time, which in the real world corresponds to overnight loans. Integrating (15) on $[t, t+h]$ we find

$$B(t+h) = B(t) e^{\int_t^{t+h} r(s)\, ds}, \tag{16}$$

which is the continuum analog of (12). Integrating (15) in the time interval $[0, t]$ we obtain the continuum analog of (14), namely

$$B(t) = B(0) \exp\left(\int_0^t r(s)\, ds \right). \tag{17}$$

Frictionless markets

Market models in financial mathematics are based on a number of simplifying assumptions which deviate, sometimes substantially, from the behavior of real markets. Among these simplifying assumptions we impose that

1. There is no bid/ask spread
2. There are no transaction costs and trades occur instantaneously
3. An investor can trade any fraction of shares
4. When a stock pays a dividend, the ex-dividend date and the payment date are the same and the stock price at this date drops by the exact same amount paid by the dividend

As seen in the previous sections real markets do not satisfy exactly these assumptions, although in some case they do it with reasonable approximation. For instance, if the investor is a professional agent managing large portfolios then the above assumptions reflect reality quite well. However they work very badly for private investors and for small portfolios. The validity of these assumptions is summarized by saying that the market has **no friction**. The idea is that, when the above assumptions hold, trading proceeds "smoothly without resistance".

In a frictionless market the portfolio process of an agent who is investing on N assets during the time interval $[0, T]$ may be defined as a function

$$\mathcal{A} : [0, T] \to \mathbb{R}^N, \quad \mathcal{A}(t) = (a_1(t), \ldots, a_N(t)),$$

i.e., by assumptions 2 and 3, the number of shares $a_i(t)$ of each single asset at time t is now allowed to be any real number and to change at any arbitrary time in the interval $[0, T]$; of course, in real market applications $a_1(t), \ldots, a_N(t)$ must be rounded to integer numbers. Portfolio processes can be added using the linear structure in $\mathbb{R}^N$, namely if $\mathcal{B} = (b_1(t), \ldots, b_N(t))$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha\mathcal{A} + \beta\mathcal{B}$ is the portfolio process

$$\alpha\mathcal{A} + \beta\mathcal{B} = (\alpha a_1(t) + \beta b_1(t), \ldots, \alpha a_N(t) + \beta b_N(t)).$$

The value at time t of the portfolio process $\mathcal{A}$ is

$$V_{\mathcal{A}}(t) = \sum_{i=1}^{N} a_i(t)\Pi^{\mathcal{U}_i}(t),$$

and clearly

$$V_{\alpha\mathcal{A}}(t) + V_{\beta\mathcal{B}}(t) = V_{\alpha\mathcal{A}+\beta\mathcal{B}}(t).$$

Moreover, thanks to assumption 3, perfect self-financing portfolio processes in frictionless markets always exist.

By assumption 1, any offer to buy/sell an asset is matched by an offer to sell/buy the asset. Of course this assumption is only reasonable when the price of the asset is *fair*. What exactly means that asset prices are fair will be the one of the main topics of study in options pricing theory.

0.2 Finite probability theory

We begin by recalling a few results on finite probability spaces. For more details on this subject, see Chapter 5 in [2].

Let $\Omega = \{\omega_1, \ldots, \omega_m\}$ be a sample space containing m elements. Let $p = (p_1, \ldots, p_m)$ be a **probability vector**, i.e.,

$$0 < p_i < 1, \text{ for all } i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1.$$

We define $p_i = \mathbb{P}(\{\omega_i\})$ to be the probability of the event $\{\omega_i\}$. If $A \subseteq \Omega$ is a non-empty event, we define the probability of A as

$$\mathbb{P}(A) = \sum_{i:\omega_i \in A} p_i = \sum_{\omega \in A} \mathbb{P}(\{\omega\}).$$

Moreover $\mathbb{P}(\emptyset) = 0$. The pair $(\Omega, \mathbb{P})$ is called a **finite probability space**. For example, given $p \in (0,1)$, the probability space

$$\Omega_N = \{H,T\}^N, \quad \mathbb{P}_p(\{\omega\}) = p^{N_H(\omega)}(1-p)^{N_T(\omega)}$$

is called the **N-coin toss probability space**. Here $N_H(\omega)$ is the number of Heads in the toss $\omega \in \Omega_N$ and $N_T(\omega) = N - N_H(\omega)$ is the number of Tails. In this probability space, tosses are independent and each toss has the same probability p to result in a head.

A **random variable** is a function $X : \Omega \to \mathbb{R}$. Y is said to be **X-measurable** if there exists a function g such that $Y = g(X)$. Two random variables X, Y are **independent** if $\mathbb{P}(X \in I, Y \in J) = \mathbb{P}(X \in I)\mathbb{P}(Y \in J)$ for every $I \subseteq \mathrm{Im}(X)$ and $J \subseteq \mathrm{Im}(Y)$, where $\mathrm{Im}(X) = \{y \in \mathbb{R} : y = X(\omega) \text{ for some } \omega \in \Omega\}$ is the image of X.

The function

$$f_X(x) = \mathbb{P}(X = x),$$

is called the **probability density function** (or probability mass function) of X. Clearly $f_X(x) = 0$ if $x \notin \mathrm{Im}(X)$. The **expectation** of X is denoted by $\mathbb{E}[X]$; it is given by

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\{\omega\})$$

and satisfies the properties in the following theorem.

Theorem 0.1. *Let X, Y be random variables, $g : \mathbb{R} \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$. The following holds:*

1. $\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$ *(linearity).*
2. *If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$.*
3. *If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$ (monotonicity).*
4. *If X, Y are independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*
5. *If $Y = g(X)$, i.e., if Y is X-measurable, then*
$$\mathbb{E}[g(X)] = \sum_{x \in \mathrm{Im}(X)} g(x) f_X(x). \tag{18}$$
6. *For any convex continuous function $f : \mathbb{R} \to \mathbb{R}$, the* **Jensen inequality** *holds:*
$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

For instance in the N-coin toss probability space consider a random variable X which is measurable with respect to N_H, i.e., $X(\omega) = g(N_H(\omega))$. Then

$$\begin{aligned}\mathbb{E}[X] &= \sum_{\omega \in \Omega_N} X(\omega)\mathbb{P}(\{\omega\}) = \sum_{\omega \in \Omega_N} g(N_H(\omega)) p^{N_H(\omega)}(1-p)^{N_T(\omega)} \\ &= \sum_{k=0}^{N} \binom{N}{k} g(k) p^k (1-p)^{N-k}, \end{aligned} \tag{19}$$

where we used that the number of N-tosses with $N_H(\omega) = k$ is given by the **binomial coefficient** $\binom{N}{k}$, for all $k = 0, \dots, N$.

The quantity

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

is called **variance** of the random variable X. The quantity

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

is called **covariance** of the random variables X, Y. We have the identities

$$\mathrm{Var}[X] = \mathrm{Cov}[X, X], \quad \mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X, Y].$$

If $\mathrm{Var}[X], \mathrm{Var}[Y]$ are both positive (i.e., if X, Y are not deterministic constants), the quantity

$$\mathrm{Corr}[X, Y] = \frac{\mathrm{Cov[X, Y]}}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \in [-1, 1]$$

is called **correlation** of X, Y. If $\mathrm{Corr}[X, Y] = 0$, the random variables X, Y are said to be **uncorrelated**. It follows by Theorem 0.1(4) that X, Y independent $\Rightarrow$ X, Y uncorrelated (while the opposite is in general not true).

The **conditional expectation** of X given Y is denoted by $\mathbb{E}[X|Y]$:

$$\mathbb{E}[X|Y](\omega) = \sum_{x \in \mathrm{Im}(X)} \mathbb{P}(X = x | Y = Y(\omega))x,$$

where $\mathbb{P}(A|B) = \mathbb{P}(B)^{-1}\mathbb{P}(A \cap B)$ is the conditional probability of the event A given the event B. The conditional expectation is a Y-measurable random variable and satisfies the following properties.

Theorem 0.2. *Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables and $\alpha, \beta \in \mathbb{R}$. Then*

1. $\mathbb{E}[\alpha X + \beta Y | Z] = \alpha\mathbb{E}[X|Z] + \beta\mathbb{E}[Y|Z]$ *(linearity).*
2. *If X is independent of Y, then $\mathbb{E}[X|Y] = \mathbb{E}[X]$.*
3. *If X is Y-measurable, then $\mathbb{E}[X|Y] = X$.*
4. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
5. *If X is Z-measurable, then $\mathbb{E}[XY|Z] = X\mathbb{E}[Y|Z]$.*
6. *If Z is Y-measurable then $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$.*

These properties remain true if the conditional expectation is taken with respect to several random variables.

A **discrete stochastic process** is a (possibly finite) sequence $\{X_0, X_1, X_2, \dots\} = \{X_n\}_{n\in\mathbb{N}}$ of random variables. We refer to the index n in X_n as **time step**. If the discrete stochastic process is finite, i.e., if it runs only for a finite number $N \geq 1$ of time steps, we shall denote it by $\{X_n\}_{n=0,\dots,N}$ and call it a N**-period process**. At each time step, a discrete stochastic process on a finite probability space is a random variable with finitely many possible values. More precisely, for all $n = 0, 1, 2, \dots$, the value x_n of X_n satisfies $x_n \in \mathrm{Im}(X_n)$. We call x_n an **admissible state** of the stochastic process. Note that x_n is an admissible state if and only if $\mathbb{P}(X_n = x_n) > 0$.

A stochastic process $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **measurable** with respect to $\{X_n\}_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$ there exists a function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $Y_n = g_n(X_0, X_2, \dots, X_n)$. If $Y_n = h_n(X_0, \dots, X_{n-1})$ for some function $h_n : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, then $\{Y_n\}_{n\in\mathbb{N}}$ is said to be **predictable** from the process $\{X_n\}_{n\in\mathbb{N}}$.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **martingale** if

$$\mathbb{E}[X_{n+1}|X_1, X_2, \dots X_n] = X_n, \quad \text{for all } n \in \mathbb{N}. \tag{20}$$

The interpretation is the following: The variables $X_0, X_1, \dots X_n$ contain the information obtained by "looking" at the stochastic process up to the time step n. For a martingale process, this information is not enough to estimate whether, in the next step, the process will raise or fall. Martingales have constant expectation, i.e., $\mathbb{E}[X_n] = \mathbb{E}[X_0]$, for all $n \in \mathbb{N}$.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **Markov chain** if it satisfies the **Markov property**:

$$\mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1}|X_1 = x_1, X_2 = x_2, \dots, X_n = x_n), \tag{21}$$

for all $n \in \mathbb{N}$ and for all admissible states $x_0 \in \mathrm{Im}(X_0), \dots, x_{n+1} \in \mathrm{Im}(X_{n+1})$ such that $\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots X_n = x_n)$ is positive[2]. The interpretation is the following: If $\{X_n\}_{n\in\mathbb{N}}$ is a Markov process, then the probability of transition from the state x_n to the state x_{n+1} does not depend on the states occupied by the process before time n. Thus Markov processes are "memoryless": at each time step they "forget" what they did earlier.

The left hand side of (21) is called the **transition probability** from the state x_n to the state x_{n+1} and is denoted also as $\mathbb{P}(x_n \to x_{n+1})$. If $\mathbb{P}(x_n \to x_{n+1})$ is independent of $n = 1, 2, \dots$, the Markov process is said to be **time homogeneous**.

Remark 0.3. If $\{X_n\}_{n\in\mathbb{N}}$ is a Markov process and $\{Y_n\}_{n\in\mathbb{N}}$ is measurable with respect to $\{X_n\}_{n\in\mathbb{N}}$, then the Markov property (21) implies

$$\mathbb{E}[Y_n|X_{n-1}] = \mathbb{E}[Y_n|X_0, \dots, X_{n-1}].$$

Remark 0.4. The Markov property and the martingale property depend on the probability measure, i.e., a stochastic process can be a martingale and/or a Markov process in one probability $\mathbb{P}$ and neither of them in another probability $\widetilde{\mathbb{P}}$.

[2]That is to say, there must be a path of the stochastic process that connects the states $x_0, \dots, x_n$.

Example: Random Walk. Consider the following stochastic process $\{X_n\}_{n=1,\dots,N}$ defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$:

$$\omega = (\gamma_1, \dots, \gamma_N) \in \Omega_N, \quad X_n(\omega) = \begin{cases} 1 & \text{if } \gamma_n = H \\ -1 & \text{if } \gamma_n = T \end{cases}.$$

The random variables $X_1, \dots, X_N$ are independent and identically distributed (**i.i.d**), namely

$$\mathbb{P}_p(X_n = 1) = p, \quad \mathbb{P}_p(X_n = -1) = 1 - p, \quad \text{for all } n = 1, \dots, N.$$

Hence

$$\mathbb{E}[X_n] = 2p - 1, \quad \mathrm{Var}[X_n] = 4p(1-p), \quad \text{for all } n = 1, \dots, N.$$

Now, for $n = 1, \dots, N$, let

$$M_0 = 0, \quad M_n = \sum_{i=1}^{n} X_i.$$

The stochastic process $\{M_n\}_{n=0,\dots,N}$ is measurable (but not predictable) with respect to the process $\{X_n\}_{n=1,\dots,N}$ and is called **(N-period) random walk**. It satisfies

$$\mathbb{E}[M_n] = n(2p - 1), \quad \text{for all } n = 0, \dots, N.$$

Moreover, since it is the sum of independent random variables, the random walk has variance given by

$$\mathrm{Var}[M_0] = 0, \quad \mathrm{Var}[M_n] = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{Var}[X_i] = 4np(1-p).$$

When $p = 1/2$, the random walk is said to be **symmetric**. In this case $\{M_n\}_{n=0,\dots,N}$ satisfies $\mathbb{E}[M_n] = 0$ and $\mathrm{Var}[M_n] = n$, $n = 0, \dots, N$. When $p \neq 1/2$, $\{M_n\}_{n=0,\dots,N}$ is called an **asymmetric** random walk, or a random walk with **drift**.

If $M_n = k$ then M_{n+1} is either $k+1$ (with probability p), or $k-1$ (with probability $1-p$). Hence we can represent the paths of the random walk by using a binomial tree, as in the following example for $N = 3$:

$$
\begin{tikzcd}[column sep=small]
 & & & M_3 = 3 \\
 & & M_2 = 2 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & M_1 = 1 \arrow[ur, "p"] \arrow[dr, "1-p"] & & M_3 = 1 \\
M_0 = 0 \arrow[ur, "p"] \arrow[dr, "1-p"] & & M_2 = 0 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & M_1 = -1 \arrow[ur, "p"] \arrow[dr, "1-p"] & & M_3 = -1 \\
 & & M_2 = -2 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & & & M_3 = -3
\end{tikzcd}
$$

By inspection we see that the admissible states of the symmetric random walk at the step n are given by

$$\mathrm{Im}(M_n) = \{-n, -n+2, -n+4, \ldots, n-2, n\} = \{2k - n,\ k = 0, \ldots, n\},$$

where k is the number of times that the random walk "goes up" up to the step n included. From this one can see that the density of M_n is given by the **binomial probability density** function

$$f_{M_n}(x) = \binom{n}{k} p^k (1-p)^{n-k} \delta(x - (2k - n)),\ k = 0, \ldots, n, \tag{22}$$

where $\delta(z) = 1$ if $z = 0$ and $\delta(z) = 0$ otherwise.

Let $m_0 = 0$, $m_1 \in \{-1, 1\} = \mathrm{Im}(M_1)$, $\ldots$, $m_N \in \{-N, -N+2, \ldots, N-2, N\} = \mathrm{Im}(M_N)$ be the admissible states at each time step. From the binomial tree of the process it is clear that there exists a path connecting $m_0, m_1, \ldots, m_N$ if and only if $m_n = m_{n-1} \pm 1$, for all $n = 1, \ldots, N$, and we have

$$
\begin{aligned}
\mathbb{P}(M_n = m_n | M_1 = m_1, \ldots, M_{n-1} = m_{n-1}) &= \mathbb{P}(M_n = m_n | M_{n-1} = m_{n-1}) \\
&= \left\{ \begin{array}{ll} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \end{array} \right. .
\end{aligned}
$$

Hence the random walk is an example of time homogeneous Markov chain.

Next we show that the *symmetric* random walk is a martingale. In fact, using the linearity of the conditional expectation we have

$$\begin{aligned}\mathbb{E}[M_n|M_1,\dots,M_{n-1}] &= \mathbb{E}[M_{n-1}+X_n|M_1,\dots,M_{n-1}]\\ &= \mathbb{E}[M_{n-1}|M_1,\dots,M_{n-1}]+\mathbb{E}[X_n|M_1,\dots,M_{n-1}].\end{aligned}$$

As M_{n-1} is measurable with respect to $M_1,\dots,M_{n-1}$, then $\mathbb{E}[M_{n-1}|M_1,\dots,M_{n-1}] = M_{n-1}$, see Theorem 0.2(3). Moreover, as X_n is independent of $M_1,\dots,M_{n-1}$, Theorem 0.2(2) gives $\mathbb{E}[X_n|M_1,\dots,M_{n-1}] = \mathbb{E}[X_n] = 0$. It follows that $\mathbb{E}[M_n|M_1,\dots,M_{n-1}] = M_{n-1}$, i.e., the symmetric random walk is a martingale. However the asymmetric random walk ($p \neq 1/2$) is *not* a martingale, as it follows by the fact that its expectation $\mathbb{E}[M_n] = n(2p-1)$ is not constant.

Generalized random walk. A random walk may be defined as any discrete stochastic process $\{M_n\}_{n\in\mathbb{N}}$ which satisfies the following properties:

- $\mathrm{Im}(M_n) = \{-n,-n+2,-n+4,\dots,n-2,n\}$, for all $n = 0,1,\dots$;
- $\{M_n\}_{n\in\mathbb{N}}$ is a time-homogeneous Markov chain;
- There exists $p \in (0,1)$ such that for $(m_{n-1},m_n) \in \mathrm{Im}(M_{n-1})\times\mathrm{Im}(M_n)$, the transition probability $\mathbb{P}(m_{n-1}\to m_n)$ is given by

$$\mathbb{P}(m_{n-1}\to m_n) = \begin{cases} p & \text{if } m_n = m_{n-1}+1\\ 1-p & \text{if } m_n = m_{n-1}-1\\ 0 & \text{otherwise}\end{cases}.$$

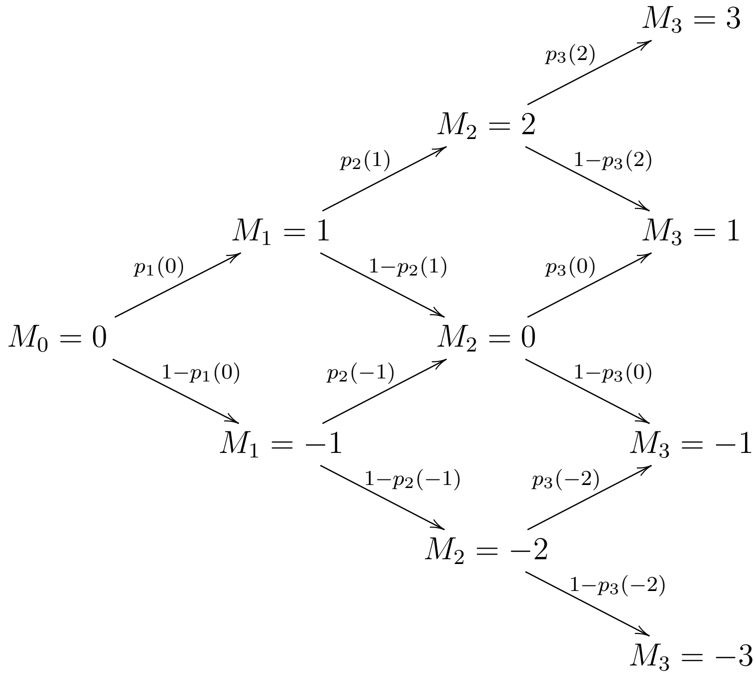
We may generalize this definition by relaxing the second and third properties as follows.

Definition 0.1. *A discrete stochastic process $\{M_n\}_{n\in\mathbb{N}}$ on a finite probability space is called a* **generalized random walk** *if it satisfies the following properties:*

1. $\mathrm{Im}(M_n) = \{-n,-n+2,-n+4,\dots,n-2,n\}$, *for all* $n = 0,1,\dots$;
2. $\{M_n\}_{n\in\mathbb{N}}$ *is a Markov chain;*
3. *For all* $n = 1,2,\dots$ *there exist* $p_n : \mathrm{Im}(M_{n-1}) \to (0,1)$ *such that*

$$\mathbb{P}(m_{n-1}\to m_n) = \begin{cases} p_n(m_{n-1}) & \textit{if } m_n = m_{n-1}+1\\ 1-p_n(m_{n-1}) & \textit{if } m_n = m_{n-1}-1\\ 0 & \textit{otherwise}\end{cases}.$$

The binomial tree of a generalized random walk will be written as in the following example:



When $p_n \equiv p$ for all $n = 1, 2, \dots$, the generalized random walk becomes the standard random walk considered before. Note carefully that the admissible states of a generalized random walk are precisely the same as for the standard random walk, but they are now attained with different probabilities. In particular the generalized random walk is no longer binomially distributed, unless of course $p_n \equiv p$ for all $n = 1, 2, \dots$.

For later purpose we give below a formula to compute the probability that the generalized random walk follows a given path. It is clear that any path in the N-period random walk is uniquely identified by a vector $x \in \{-1, 1\}^N$, i.e., a N-dimensional vector where each component is either -1 or 1. More precisely, the path of the random walk corresponding to $x \in \{-1, 1\}^N$ it the unique path satisfying $M_0 = 0$ and $M_i = M_{i-1} + x_i$, $i = 1, \dots, N$.

Theorem 0.3. *Let $x \in \{-1, 1\}^N$ and set $x_0 = 0$. The probability $\mathbb{P}(x)$ that the generalized random walk follows the path x is given by*

$$\mathbb{P}(x) = \prod_{t=1}^{N} \left[-\min(x_t, 0) + x_t p_t \left(\sum_{j=0}^{t-1} x_j \right) \right]. \tag{23}$$

The previous theorem can be easily proved by induction, but here we limit ourselves to consider one example of application of (23). In the 3-period model consider the path $x = (-1, -1, 1)$. Then according to the previous theorem

$$\begin{aligned} \mathbb{P}((-1,-1,1)) &= (-\min(-1,0) + (-1)p_1(0))(-\min(-1,0) + (-1)p_2(0-1)) \\ &\quad \times (-\min(1,0) + (1)p_3(0-1-1)) = (1-p_1(0))(1-p_2(-1))p_3(-2). \end{aligned}$$

That this formula is correct is easily seen in the binomial tree above.

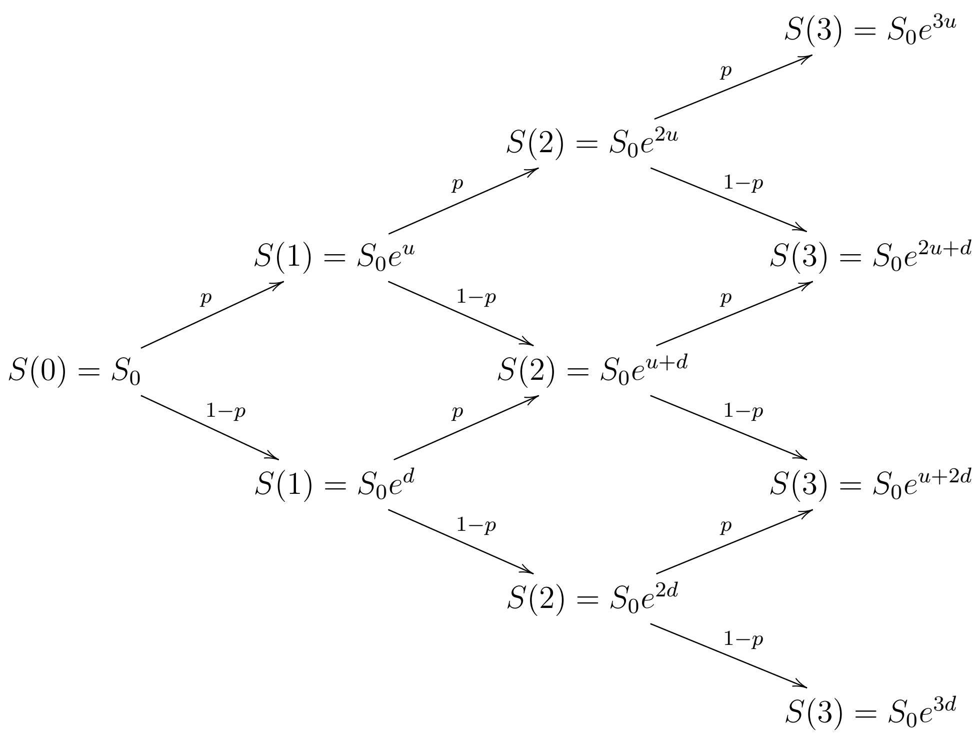
The generalized random walk will be used in Section 0.4 to introduce a generalized binomial model in which the risk-free asset is a stochastic process. The standard binomial model, in which the risk-free rate is assumed to be constant, is reviewed in the next section.

0.3 The binomial model with constant risk-free rate

Given $0 < p < 1$, $S_0 > 0$ and $u > d$, the **binomial stock price** at time t is given by $S(0) = S_0$ and

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases}, \qquad \text{for } t = 1, \dots, N. \tag{24}$$

If $S(t) = S(t-1)e^u$ we say that the stock price goes up at time t, while if $S(t) = S(t-1)e^d$ we say that it goes down at time t (although this terminology is strictly correct only when $u > 0$ and $d < 0$). For instance, for $N = 3$ the binomial stock can be represented as in the following recombining binomial tree:



The possible stock prices at time t belong to the set

$$\mathrm{Im}(S(t)) = \{S_0e^{ku+(t-k)d},\ k(t) = 0, \dots, t\},$$

where k is the number of times that the price goes up up to and including time t. It follows that there are $t+1$ possible prices at time t and so the number of nodes in the binomial tree grows linearly in time. Moreover the stock price is binomially distributed, namely

$$f_{S(t)}(x) = \binom{t}{k} p^k (1-p)^{t-k} \delta(x - S_0 e^{ku+(t-k)d}),\ k = 0, \dots, t. \tag{25}$$

The binomial stock price can be interpreted as a stochastic process defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. To see this, consider the following i.i.d. random variables

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \left\{ \begin{array}{ll} 1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } H \\ -1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } T \end{array} \right. , \quad t = 1, \dots, N.$$

We can rewrite (24) as $S(t) = S(t-1)\exp[(u+d)/2 + (u-d)X_t/2]$, which upon iteration leads to

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_t \right], \quad M_t = X_1 + \dots + X_t,\ t = 1, \dots, N. \tag{26}$$

Hence $S(t) : \Omega_N \to \mathbb{R}$ and therefore $\{S(t)\}_{t=0,\dots,N}$ is a N-period stochastic process on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. In this context, $\mathbb{P}_p$ is called **physical** (or **real-world**) **probability measure**, to distinguish it from the martingale (or risk-neutral) probability introduced below. Letting $M_0 = 0$, we have that $\{M_t\}_{t=0,\dots,N}$ is a random walk (which is asymmetric for $p \neq 1/2$). It follows that $\{S(t)\}_{t=0,\dots,N}$ is measurable, but not predictable, with respect to $\{M_t\}_{t=0,\dots,N}$. For each $\omega \in \Omega_N$, the vector $(S(0), S(1,\omega), \dots, S(N,\omega))$ is called a **path** of the binomial stock price.

A **binomial market** is a market that consists of one stock with price given by (26), and a **risk-free asset** with value $B(t)$ at time $t = 1, \dots, N$. In the standard binomial model it is assumed that $B(t)$ is a deterministic function of time with constant interest rate, namely

$$r = \log B(t+1) - \log B(t), \quad \text{or} \quad R = \frac{B(t+1) - B(t)}{B(t)}.$$

It follows that the value of the risk-free asset at time t can be written in either of the two forms

$$B(t) = B_0 e^{rt}, \quad \text{or} \quad B(t) = B_0(1+R)^t, \quad t = 1, \dots, N,$$

where B_0 is the initial value of the risk-free asset. We shall refer to R as the **discretely compounded risk-free rate** and to r as the **continuously compounded risk-free rate** (although the latter terminology is only strictly correct in the time continuum limit, i.e., when we let the length of the time step tends to zero). Note also that

$$r = \log(1+R). \tag{27}$$

As r and R are small, then $r \approx R$.

Remark 0.5. In [2] only the continuously compounded risk-free rate r was used. Here we introduced the discretely compounded risk-free rate R as well because it will be used in Section 0.4 to formulate a generalized binomial model with stochastic risk-free rate.

The quantity

$$S^*(t) = e^{-rt}S(t), \quad \text{or equivalently} \quad S^*(t) = \frac{S(t)}{(1+R)^t},$$

is called the **discounted price** of the stock (at time $t = 0$).

In the following we denote by $\mathbb{E}_p$ the (possibly conditional) expectation in the probability space $(\Omega_N, \mathbb{P}_p)$.

Theorem 0.4. *If $r \notin (d,u)$, there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale. For $r \in (d,u)$, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where*

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

Proof. By definition, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale if and only if

$$\mathbb{E}_p[S^*(t)|S^*(0),\dots,S^*(t-1)] = S^*(t-1), \quad \text{for all } t = 1,\dots,N.$$

Taking the expectation conditional to $S^*(0),\dots,S^*(t-1)$ is clearly the same as taking the expectation conditional to $S(0),\dots,S(t-1)$, hence the above equation is equivalent to

$$\mathbb{E}_p[S(t)|S(0),\dots,S(t-1)] = e^r S(t-1), \quad \text{for all } t = 1,\dots,N, \tag{28}$$

where we canceled out a factor e^{-rt} in both sides of the equation. Moreover

$$\begin{aligned}\mathbb{E}_p[S(t)|S(0),\dots,S(t-1)] &= \mathbb{E}_p[\frac{S(t)}{S(t-1)}S(t-1)|S(0),\dots,S(t-1)]\\ &= S(t-1)\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0),\dots,S(t-1)],\end{aligned}$$

where we used that $S(t-1)$ is measurable with respect to the conditioning variables and thus it can be taken out from the conditional expectation (see property 5 in Theorem 0.2). As

$$S(t)/S(t-1) = \begin{cases} e^u & \text{with prob. } p \\ e^d & \text{with prob. } 1-p \end{cases}$$

is independent of $S(0),\dots,S(t-1)$, then by Theorem 0.2(2) we have

$$\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0),\dots,S(t-1)] = \mathbb{E}_p[\frac{S(t)}{S(t-1)}] = e^u p + e^d(1-p).$$

Hence (28) holds if and only if $e^u p + e^d(1-p) = e^r$. Solving in $p \in (0,1)$ we find $p = q$ and the condition $0 < q < 1$ is then equivalent to $r \in (d,u)$. □

Due to Theorem 0.4, $\mathbb{P}_q$ is called **martingale probability measure**. Moreover, since martingales have constant expectation, then

$$\mathbb{E}_q[S(t)] = S_0 e^{rt}. \tag{29}$$

Thus in the martingale probability measure one expects the same return on the stock as on the risk-free asset. For this reason, $\mathbb{P}_q$ is also called **risk-neutral probability**.

Self-financing portfolios

A **portfolio process** in a binomial market is a stochastic process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ such that, for $t = 1, \dots, N$, $(h_S(t), h_B(t))$ corresponds to the portfolio position (number of shares) on the stock and the risk-free asset held in the interval $(t-1, t]$. A positive number of shares corresponds to a long position on the asset, while a negative number of shares corresponds to a short position. As portfolio positions held for one instant of time only are meaningless, we use the convention $h_S(0) = h_S(1), h_B(0) = h_B(1)$, that is to say, $h_S(1), h_B(1)$ is the portfolio position in the *closed* interval $[0, 1]$. We always assume that the portfolio process is predictable from $\{S(t)\}_{t=0,\dots,N}$, i.e., there exists functions $H_t : (0, \infty)^t \to \mathbb{R}^2$ such that $(h_S(t), h_B(t)) = H_t(S(0), \dots, S(t-1))$. Thus the decision on which position the investor should take in the interval $(t-1, t]$ depends only on the information available at time $t-1$. The **value** of the portfolio process is the stochastic process $\{V(t)\}_{t=0,\dots,N}$ given by

$$V(t) = h_B(t)B(t) + h_S(t)S(t), \quad t = 0, \dots, N. \tag{30}$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ is said to be **self-financing** if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1), \quad t = 1, \dots, N, \tag{31}$$

while it is said to generate the **cash flow** $C(t-1)$ if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1) + C(t-1), \quad t = 1, \dots, N. \tag{32}$$

Recall that $C(t) > 0$ corresponds to cash withdrawn from the portfolio at time t while $C(t) < 0$ corresponds to cash added to the portfolio at time t. The self-financing property means that no cash is ever added or withdrawn from the portfolio.

Theorem 0.5. *Let $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ be a self-financing predictable portfolio process with value $\{V(t)\}_{t=0,\dots,N}$. Then the discounted portfolio value $\{V^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability measure. Moreover the following identity holds:*

$$V^*(t) = \mathbb{E}_q[V^*(N)|S(0), \dots, S(t)], \quad t = 0, \dots, N. \tag{33}$$

Proof. The martingale claim is

$$\mathbb{E}_q[V^*(t)|V^*(0), \dots, V^*(t-1)] = V^*(t-1).$$

We now show that this follows by

$$\mathbb{E}_q[V^*(t)|S(0),\dots,S(t-1)] = V^*(t-1). \tag{34}$$

In fact, computing the expectation of (34) conditional to $V^*(0),\dots,V^*(t-1)$, we obtain

$$\begin{aligned} V^*(t-1) &= \mathbb{E}_q[V^*(t-1)|V^*(0),\dots,V^*(t-1)] \\ &= \mathbb{E}_q\big[\mathbb{E}_q[V^*(t)|S(0),\dots,S(t-1)]|V^*(0),\dots,V^*(t-1)\big] \\ &= \mathbb{E}_q[V^*(t)|V^*(0),\dots,V^*(t-1)], \end{aligned}$$

where we have used property 3 of Theorem 0.2 in the first equality and property 6 in the last equality. The latter is possible because $V^*(t)$ is measurable with respect to $S(0),\dots,S(t)$. Now we claim that (34) also implies the formula (33). We argue by backward induction. Letting $t = N$ in (34) we see that (33) holds at $t = N-1$. Assume now that (33) holds at time $t+1$, i.e.,

$$V^*(t+1) = \mathbb{E}_q[V^*(N)|S(0),\dots,S(t+1)].$$

Taking the expectation conditional to $S(0),\dots,S(t)$ we have, by (34),

$$\begin{aligned} V^*(t) &= \mathbb{E}_q[V^*(t+1)|S(0),\dots,S(t)] = \mathbb{E}_q\Big[\mathbb{E}_q[V^*(N)|S(0),\dots,S(t+1)]|S(0),\dots,S(t)\Big] \\ &= \mathbb{E}_q[V^*(N)|S(0),\dots,S(t)]. \end{aligned}$$

Hence (33) holds at time t and so (34) $\Rightarrow$ (33), as claimed. Finally we prove (34). As $B(t) = B(t-1)e^r$, (31) gives

$$h_B(t)B(t) = e^r V(t-1) - h_S(t)S(t-1)e^r.$$

Replacing in (30) we find

$$V(t) = e^r V(t-1) + h_S(t)[S(t) - S(t-1)e^r].$$

Taking the expectation conditional to $S(0),\dots,S(t-1)$ we obtain

$$\begin{aligned} \mathbb{E}_q[V(t)|S(0),\dots S(t-1)] &= e^r\mathbb{E}_q[V(t-1)|S(0),\dots,S(t-1)] \\ &\quad + \mathbb{E}_q[h_S(t)(S(t)-S(t-1)e^r)|S(0),\dots,S(t-1)]. \end{aligned} \tag{35}$$

As $V(t-1)$ and $h_S(t)$ are measurable with respect to the conditioning variables we have $\mathbb{E}_q[V(t-1)|S(0),\dots,S(t-1)] = V(t-1)$, as well as

$$\begin{aligned} &\mathbb{E}_q[h_S(t)(S(t)-S(t-1)e^r)|S(0),\dots,S(t-1)] \\ &= h_S(t)\mathbb{E}_q[S(t)-S(t-1)e^r|S(0),\dots,S(t-1)] \\ &= h_S(t)\Big(\mathbb{E}_q[S(t)|S(0),\dots,S(t-1)] - S(t-1)e^r\Big) = 0, \end{aligned}$$

where in the last step we used that $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability. Going back to (35) we obtain

$$\mathbb{E}_q[V(t)|S(0),\dots S(t-1)] = e^r V(t-1),$$

which is the same as (34). $\square$

Arbitrage portfolios

A portfolio process $\{(h_S(t), h_B(t)\}_{t=0,\dots,N}$ invested in the binomial market is called an **arbitrage portfolio process** if it is predictable and if its value $V(t)$ satisfies

1) $V(0) = 0$;

2) $V(N, \omega) \geq 0$, for all $\omega \in \Omega_N$;

3) There exists $\omega_* \in \Omega_N$ such that $V(N, \omega_*) > 0$.

Theorem 0.6. *Assume $d < r < u$, i.e., assume the existence of a risk-neutral probability measure for the binomial market. Then the binomial market is free of self-financing arbitrages.*

Proof. Assume that $\{h_S(t), h_B(t)\}_{t=0,\dots,N}$ is a self-financing arbitrage portfolio process. Then $V(0) = V^*(0) = 0$ and since martingales have constant expectation then $\mathbb{E}_q[V^*(t)] = 0$, for all $t = 0, 1, \dots, N$. As $V(N) \geq 0$, then $V^*(N) \geq 0$ and Theorem 0.1(2) entails $V^*(N, \omega) = 0$ for any sample $\omega \in \Omega_N$. Hence $V(N, \omega) = 0$, for all $\omega \in \Omega_N$, contradicting the assumption that the portfolio is an arbitrage. $\square$

Remark 0.6. As shown in [2], the existence of a risk-neutral probability measure in not only sufficient but also necessary for the absence of self-financing arbitrages in the binomial market. More precisely, if $r \notin (d, u)$ one can construct self-financing arbitrage portfolios in the market. Hence the binomial market is free of self-financing arbitrages if and only if it admits a risk-neutral probability measure. The latter result is valid for any discrete (or even continuum) market model and is known as the **first fundamental theorem of asset pricing**.

Risk neutral pricing formula for European derivatives in the binomial model

Let $Y : \Omega_N \to \mathbb{R}$ be a random variable and consider the European-style derivative with pay-off Y at maturity time $T = N$. This means that the derivative can only be exercised at time $t = N$. For standard European derivatives Y is a deterministic function of $S(N)$, while for non-standard derivatives Y is a deterministic function of $S(0), \dots, S(N)$. Let $\Pi_Y(t)$ be the binomial fair price of the derivative a time t. By definition, $\Pi_Y(t)$ equals the value $V(t)$ of self-financing, hedging portfolios. In particular, $\Pi_Y(t)$ is a random variable and so $\{\Pi_Y(t)\}_{t=0,\dots,N}$ is a stochastic process. Using the hedging condition $V(N) = Y$ (which means $V(N, \omega) = Y(\omega)$, for all $\omega \in \Omega_N$)) and (33), we have the following formula for the fair price at time t of the financial derivative:

$$\Pi_Y(t) = e^{-r(N-t)} \mathbb{E}_q[Y | S(0), \dots, S(t)]. \tag{36}$$

Equation (36) is known as **risk-neutral pricing formula** and it is the cornerstone of options pricing theory. It holds not only for the binomial model but for any discrete—or even continuum —pricing model for financial derivatives. It is used for standard as well as non-standard European derivatives. In the special case $t = 0$, (36) reduces to

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y]. \tag{37}$$

Remark 0.7. We may interpret (37) as follows: the current (at time $t = 0$) fair value of the derivative is our expectation on the future payment of the derivative (the pay-off) expressed in terms of the future value of money (discounted pay-off $Y^* = e^{-rN}Y$). The expectation has to be taken *with respect to the martingale probability measure*, i.e., ignoring any (subjective or illegal[3]) estimate on future movements of the stock price (except for the loss in value due to the time-devaluation of money).

Example. Consider a 2-period binomial model with the following parameters

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad r = 0, \quad p \in (0,1).$$

Assume further that $S_0 = 36$. Consider the European derivative with pay-off

$$Y = (S(2) - 28)_+ - 2(S(2) - 32)_+ + (S(2) - 36)_+$$

and time of maturity $T = 2$. According to (37), the fair value of the derivative at $t = 0$ is

$$\Pi_Y(0) = e^{-2r}\mathbb{E}_q[Y] = \mathbb{E}_q[(S(2) - 28)_+] - 2\mathbb{E}_q[(S(2) - 32)_+] + \mathbb{E}_q[(S(2) - 36)_+].$$

By the market parameters we find $q = 1/2$. Hence the distribution of $S(2)$ in the risk-neutral probability measure is

$$\mathbb{P}_q(S(2) = s) = \begin{cases} 1/4 & \text{if } s = 16 \text{ of } s = 64 \\ 1/2 & \text{if } s = 32 \\ 0 & \text{otherwise} \end{cases}.$$

It follows that

$$\mathbb{E}_q[(S(2) - 28)_+] = 11, \quad \mathbb{E}_q[(S(2) - 32)_+] = 8, \quad \mathbb{E}_q[(S(2) - 36)_+] = 7,$$

hence $\Pi_Y(0) = 2$.

By definition of expectation in the N-coin toss probability space, see (19), the risk-neutral pricing formula (37) for the standard European derivative with pay-off $Y = g(S(N))$ and maturity $T = N$ takes the explicit form

$$\Pi_Y(0) = e^{-rN}\sum_{k=0}^{N}\binom{N}{k}q^k(1-q)^k g(S_0 e^{ku+(N-k)d}).$$

[3]Trading in the market using privileged information is a crime (**insider trading**).

However this formula is not very convenient for numerical computations, because the binomial coefficient $\binom{N}{k}$ will reach very large values for even a relative small number of steps (e.g., $\binom{50}{25}$ is of order 10^{14}). A much more convenient way to compute numerically the binomial price of standard European derivatives is by using the recurrence formula $\Pi_Y(N) = Y$ and

$$\Pi_Y(t) = e^{-r}(q\Pi_Y^u(t+1) + (1-q)\Pi_Y^d(t+1)), \quad t = 0, \ldots, N-1, \tag{38}$$

where $\Pi_Y^u(t)$ is the binomial price of the derivative at time t assuming that the stock price goes up at time t, i.e.,

$$\Pi_Y^u(t) = e^{-r(N-t)}\mathbb{E}_q[Y|S(0), \ldots, S(t-1), S(t) = S(t-1)e^u]$$

and similarly one defines $\Pi_Y^d(t)$ by replacing "up" with "down". The formula (38) follows immediately by (36) and the definition of conditional expectation.

Remark 0.8. It can be shown that any European derivative in the binomial market can be hedged by a self-financing portfolio invested in the underlying stock and the risk-free asset, see [2]. For this reason the binomial market is called a **complete market**. In fact, the **second fundamental theorem of asset pricing** states that market completeness is equivalent to the uniqueness of the risk-neutral probability measure. An arbitrage free market is said to be **incomplete** if the risk-neutral measure is not unique. When the market is incomplete the price of European derivatives is not uniquely defined and moreover there exist European derivatives which cannot be hedged by self-financing portfolios. An example of incomplete market is the trinomial model discussed in the project in Chapter 1.

Implementation of the binomial model

For real world applications the binomial model must be properly rescaled in time. Precisely, let $T > 0$ be the maturity of a European derivative and consider the uniform partition of the interval $[0, T]$ with size $h > 0$:

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \ldots, N.$$

The binomial stock price on the given partition is given by $S(0) = S_0 > 0$ and

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, & \text{with probability } p, \\ S(t_{i-1})e^d, & \text{with probability } 1-p, \end{cases} \qquad i = 1, \ldots, N,$$

while

$$B(t_i) = B_0 e^{rhi}.$$

The **instantaneous mean of log-return** and the **instantaneous variance** of the binomial stock price are defined respectively by

$$\alpha = \frac{1}{h}\mathbb{E}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{1}{h}[pu + (1-p)d],$$
$$\sigma^2 = \frac{1}{h}\mathrm{Var}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{(u-d)^2}{h}p(1-p),$$

while σ itself is called **instantaneous volatility**. The parameters α, σ are constant in the standard binomial model and are computed using the physical probability (and *not* the risk-neutral probability). Inverting the equations above we obtain

$$u = \alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}, \quad d = \alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}. \tag{39}$$

In the applications of the binomial model it is customary to give the parameters α, σ and then compute u, d using (39). The risk-neutral probability then becomes

$$q = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}. \tag{40}$$

The binomial model is trustworthy only for h very small compared to T (i.e., $N >> 1$).

The Matlab Code 1 defines a function $\texttt{EuroZeroBin}(\texttt{g}, \texttt{T}, \texttt{s}, \texttt{alpha}, \texttt{sigma}, \texttt{r}, \texttt{p}, \texttt{N})$ that computes the initial price of the standard European derivative with pay-off $Y = g(S(T))$ using (38). The variable s is the initial price S_0 of the stock. The function also checks that $q \in (0, 1)$, i.e., that the risk-neutral probability is well defined (and thus the market is free of self-financing arbitrages). If not a message appears which asks to increase the number of steps N.

For instance, upon running the command

$$\texttt{Pzero} = \texttt{EuroZeroBin}(@(\texttt{x})\ \texttt{max}(\texttt{x} - \texttt{11}, \texttt{0}), \texttt{1/3}, \texttt{10}, \texttt{0}, \texttt{0.5}, \texttt{0.01}, \texttt{1/2}, \texttt{10000})$$

we get the output

$$\texttt{Pzero} = \texttt{0.7813},$$

which is the (binomial) price at time $t = 0$ of a European call with strike $K = 11$ and maturity $T = 1/3$ years (4 months) on a stock which at $t = 0$ is priced 10 and which has volatility $\sigma = 0.5$ (i.e., 50%) and zero mean of log-return ($\alpha = 0$). The (annual) risk free rate is $r = 0.01$ (i.e., 1%). Moreover $p = 1/2$ and $N = 10000$. Prices are expressed in an arbitrary unit of currency (e.g., dollars) and the final result has been truncated at the fourth decimal digit.

As shown in [2], the binomial price of the derivative is very weakly dependent on the parameter $\alpha \in \mathbb{R}$ and $p \in (0, 1)$ (provided N is sufficiently large, say $N \approx 10000$). Hence one normally chooses $\alpha = 0$ and $p = 1/2$ in the implementation of the binomial model.

0.4 A binomial model with stochastic risk-free rate

In this section we present a generalization of the binomial model in which the risk-free rate is promoted to a stochastic process. Considering markets with random interest rates is important for the valuation of long maturity contracts. The binomial model with stochastic

```
function Pzero=EuroZeroBin(g,T,s,alpha,sigma,r,p,N)
h=T/N;
u=alpha*h+sigma*sqrt(h)*sqrt((1-p)/p);
d=alpha*h-sigma*sqrt(h)*sqrt(p/(1-p));
qu=(exp(r*h)-exp(d))/(exp(u)-exp(d));
qd=1-qu;
if (qu<0 || qd<0)
display('Error:  the market is not arbitrage free.  Increase the
value of N');
Pzero=0;
return
end
S=zeros(N+1,1);
P=zeros(N+1);
S=s*exp((N-[0:N])*u+[0:N]*d).';
P(:,N+1)=g(S);
for j=N:-1:1
for i=1:j
P(i,j)=exp(-r*h)*(qu*P(i,j+1)+qd*P(i+1,j+1));
end
end
```

Code 1: Matlab function to compute the binomial price of European options at time $t = 0$.

interest rate presented in this section will be used in the project in Chapter 2 on forward and futures contracts.

Let $\{M_t\}_{t=0,\dots,N}$ be a N-period *generalized* random walk with transition probabilities

$$\mathbb{P}(m_{t-1} \to m_t) = \begin{cases} p_t(m_{t-1}) & \text{if } m_t = m_{t-1} + 1 \\ 1 - p_t(m_{t-1}) & \text{if } m_t = m_{t-1} - 1 \\ 0 & \text{otherwise} \end{cases} . \tag{41}$$

We consider a binomial market consisting of a stock with price

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_t\right], \quad t = 0, 1, 2, \dots N, \tag{42a}$$

together with a risk-free asset with value

$$B(t) = B(t-1)(1 + R(t-1)), \quad t = 1, 2, \dots N,$$

where $\{R(t)\}_{t=0,\dots,N-1}$, $R(t) > -1$, is the discretely compounded interest rate process. By iterating the previous equation it follows that

$$B(t) = B_0 \prod_{k=0}^{t-1}(1 + R(k)), \tag{42b}$$

where $B_0 > 0$ is the initial value of the risk-free asset. As $1 + R(t) > 0$, then $B(t) > 0$ for all $t = 0, \dots, N$.

Remark 0.9. We may also introduce the continuously compounded risk-free rate process $\{r(t)\}_{t=0,\dots,N-1}$ through the formula $r(t) = \log(1 + R(t))$, $t = 0, \dots, N-1$. In terms of $r(t)$, we can write (42b) as $B(t) = B_0 \exp\sum_{k=0}^{t-1} r(k)$, which in the case $r(t) = r$ =constant reduces to the formula $B(t) = B_0 e^{rt}$ used in Section 0.3. For the study of the binomial model with stochastic risk-free rate it is preferable to work with the process $\{R(t)\}_{t=0,\dots,N-1}$.

Remark 0.10. Note carefully that the possible stock prices at time t in the market (42) are the same as in the standard binomial market model, but now they are attained with a different probability which depends on time and on the price of the stock at time $t-1$. In particular, in the present model $S(t)$ is *not* binomially distributed as it is in the standard binomial model, see (25), unless of course we choose $p_n \equiv p$ for all $n = 1, 2, \dots$.

In the following we assume that the risk-free process $\{R(t)\}_{t=0,\dots,N-1}$ is measurable with respect to the generalized random walk $\{M_t\}_{t=0,\dots,N}$. In particular the stochastic process $\{M_t\}_{t=0,\dots,N}$ completely defines the state of the binomial market.

The discounted value (at time $t = 0$) of the stock in the market (42) is defined as $S^*(t) = \frac{B_0}{B(t)} S(t)$, that is

$$S^*(t) = D(t)S(t) = \frac{S(t)}{(1 + R(0))(1 + R(1)) \dots (1 + R(t-1))}, \tag{43}$$

where

$$D(0) = 1, \quad D(t) = \prod_{k=0}^{t-1} (1 + R(k))^{-1}, \quad t = 1, \dots, N \tag{44}$$

is the **discount process**. The market (42) is arbitrage free if there exist transition probabilities (41) which make the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ a martingale; if this martingale probability is unique, the market is complete. We discuss below one example.

The Ho-Lee model

The literature abounds of stochastic models for the risk-free rate, see [1]. In this section we shall study the (discrete) **Ho-Lee model**:

$$R(t) = a(t) + b(t)M_t, \quad \text{where } a(t) \in \mathbb{R} \text{ and } b(t) > 0, \quad t = 0, 1, 2, \dots, N-1. \tag{45}$$

Since the minimum value of M_t is $-t$, then the condition $R(t) > -1$ is satisfied along all paths if and only if

$$a(t) > b(t)t - 1, \tag{46}$$

which will be assumed from now on. Our purpose is to prove that the market (42), with the risk-free rate given by the Ho-Lee model, is complete under simple conditions on the market parameters. We have the following analog of Theorem 0.4.

Theorem 0.7. *The market* (42) *admits a martingale probability measure if and only if the functions $a(t), b(t)$ are such that*

$$e^d < 1 + a(t) - b(t)t, \quad \textit{and} \quad 1 + a(t) + b(t)t < e^u, \tag{47}$$

for all $t = 0, 1, \dots, N-1$. Moreover, when it exists, the martingale probability measure is unique and it is given by $p_t(k) = q_t(k)$, where

$$q_t(k) = \frac{1 + a(t-1) + b(t-1)k - e^d}{e^u - e^d}, \tag{48}$$

where $t = 1, \dots, N, k \in \mathrm{Im}(M_{t-1}) = \{-t+1, -t+3, \dots, t-3, t-1\}$. Thus, under the conditions (47)*, the market* (42) *is complete (see Remark 0.8).*

Proof. As $\{S^*(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$, it suffices to prove that

$$\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] = S^*(t-1), \quad t = 1, 2, \dots, N. \tag{49}$$

As $R(t)$ is measurable with respect to M_t, the discount process can be taken out from the conditional expectation in the left hand side of (49), hence

$$\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] = \frac{\mathbb{E}[S(t)|M_0, \dots, M_{t-1}]}{(1 + R(0)) \dots (1 + R(t-1))} = \frac{\mathbb{E}[S(t)|M_{t-1}]}{(1 + R(0)) \dots (1 + R(t-1))},$$

where for the second equality we use that $\{S(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$ and that $\{M_t\}_{t=0,\dots,N}$ is a Markov process (see Remark 0.3). Writing $S(t) = \frac{S(t)}{S(t-1)}S(t-1)$ and using that $S(t-1)$ is M_{t-1}-measurable we obtain

$$\mathbb{E}[S^*(t)|M_0,\dots,M_{t-1}] = \frac{S(t-1)}{(1+R(0))\dots(1+R(t-1))}\mathbb{E}[\frac{S(t)}{S(t-1)}|M_{t-1}].$$

Next we use

$$\frac{S(t-1)}{(1+R(0))\dots(1+R(t-1))} = \frac{S^*(t-1)}{1+R(t-1)}, \quad \frac{S(t)}{S(t-1)} = e^{\frac{u+d}{2}}e^{\frac{u-d}{2}(M_t-M_{t-1})}.$$

According to (41), the increments of the process $\{M_t\}_{t=0,\dots,N}$ satisfy

$$\mathbb{P}(M_t - M_{t-1} = 1|M_{t-1} = k) = p_t(k), \quad \mathbb{P}(M_t - M_{t-1} = -1|M_{t-1} = k) = 1 - p_t(k).$$

Hence

$$\begin{aligned}\mathbb{E}[S^*(t)|M_0,\dots,M_{t-1}] &= \frac{S^*(t-1)}{1+R(t-1)}\mathbb{E}[e^{\frac{u+d}{2}}e^{\frac{u-d}{2}(M_t-M_{t-1})}|M_{t-1}] \\ &= S^*(t-1)\frac{e^{\frac{u+d}{2}}}{1+a(t-1)+b(t-1)k}\left(e^{\frac{u-d}{2}}p_t(k) + e^{-\frac{u-d}{2}}(1-p_t(k))\right).\end{aligned}$$

Thus in order for $p_t(k)$ to be a martingale probability it must hold that

$$\frac{e^{\frac{u+d}{2}}}{1+a(t-1)+b(t-1)k}\left(e^{\frac{u-d}{2}}p_t(k) + e^{-\frac{u-d}{2}}(1-p_t(k))\right) = 1.$$

Solving the latter equation we find $p_t(k) = q_t(k)$, where $q_t(k)$ is given by (48). Moreover $0 < q_t(k) < 1$ holds if and only if (47) are satisfied, which concludes the proof of the theorem. □

Remark 0.11. It is clear that the transition probabilities are constant if and only if $b \equiv 0$ and $a(t) = a(0)$, for all $t = 1,\dots,N-1$, i.e., if and only if the risk-free rate is a deterministic constant, in which case we go back to the standard binomial model.

Remark 0.12. While Theorem 0.7 gives a unique martingale probability, it says (of course) nothing about the physical probability. In the applications of the Ho-Lee model it is also assumed that the *physical* transition probabilities are constants and given by $p_t \equiv p = 1/2$; in particular, $\{M_t\}_{t=0,\dots,N}$ is a standard symmetric random walk in the physical probability. (In the time-continuum Ho-Lee model, which is the one actually used in the applications, $\{M_t\}_{t=0,\dots,N}$ is replaced by a Brownian motion, which is the time-continuum limit of the standard symmetric random walk, see Section 0.5.) The assumed distribution of stock prices and interest rates in the physical probability is important for the calibration of the model.

Example. Let a_0, b_0 be constants such that $b_0 > 0$. When

$$a(t) = a(0) := a_0, \quad b(t) = \frac{b_0}{t}, \quad t = 1, \dots, N-1, \tag{50}$$

the conditions (46)-(47) become

$$a_0 > b_0 - 1, \quad e^d < 1 + a_0 - b_0, \quad e^u > 1 + a_0 + b_0 \tag{51}$$

and the martingale transition probabilities read

$$q_1(0) = \frac{1 + a_0 - e^d}{e^u - e^d}, \quad q_t(k) = \frac{1 + a_0 + \frac{b_0 k}{t-1} - e^d}{e^u - e^d}, \tag{52}$$

for $t = 2, \dots, N$, $k \in \{-t+1, -t+3, \dots, t-1\}$.

Remark. Note that, in the last example, $a_0 = R(0)$ is the initial value of the risk-free rate, while $b_0 = b(1)$ is the volatility of the risk-free rate in the first time period. This remark should help to figure out what are reasonable values for these two parameters, which is important for the Matlab task in the project in Chapter 2.

European derivatives on the stock

Next we study the problem of pricing European derivatives in the market (42).

Definition 0.2. *Assume that the market* (42) *is complete (e.g., the risk-free rate is given by the Ho-Lee model and the conditions* (47) *are verified). Consider a European derivative with maturity $T = N$ and pay-off Y which is measurable with respect to $M_0, \dots, M_N$ (e.g., $Y = g(S(N))$ for a standard European derivative on the stock). The risk-neutral price of the derivative is given by*

$$\Pi_Y(t) = D(t)^{-1} \widetilde{\mathbb{E}}[D(T)Y | M_0, \dots, M_t], \quad t = 0, \dots, T, \tag{53}$$

where $\widetilde{\mathbb{E}}$ denotes the (conditional) expectation in the martingale probability measure. In particular $\Pi_Y(T) = Y$ and

$$\Pi_Y(0) = \widetilde{\mathbb{E}}[D(T)Y] = \widetilde{\mathbb{E}}[\prod_{k=0}^{T-1} (1 + R(k))^{-1} Y]. \tag{54}$$

Remark 0.13. Note that even in the case of standard financial derivatives, the discounted pay-off is now path dependent, which makes the (numerical) computation of the expectation in (54) considerably more difficult than in the standard binomial model.

For example, the **zero coupon bond** (ZCB) with **face value** K and maturity T is the European style derivative that promises to pay K at time T. It follows by (53) that the value of the ZCB at time t is given by

$$B_K(t, T) = K D(t)^{-1} \widetilde{\mathbb{E}}[D(T) | M_0, \dots, M_t] \quad t = 0, \dots, T = N.$$

When $K = 1$ we denote $B_K(t,T)$ simply as $B(t,T)$. Clearly, $B_K(t,T) = KB(t,T)$.

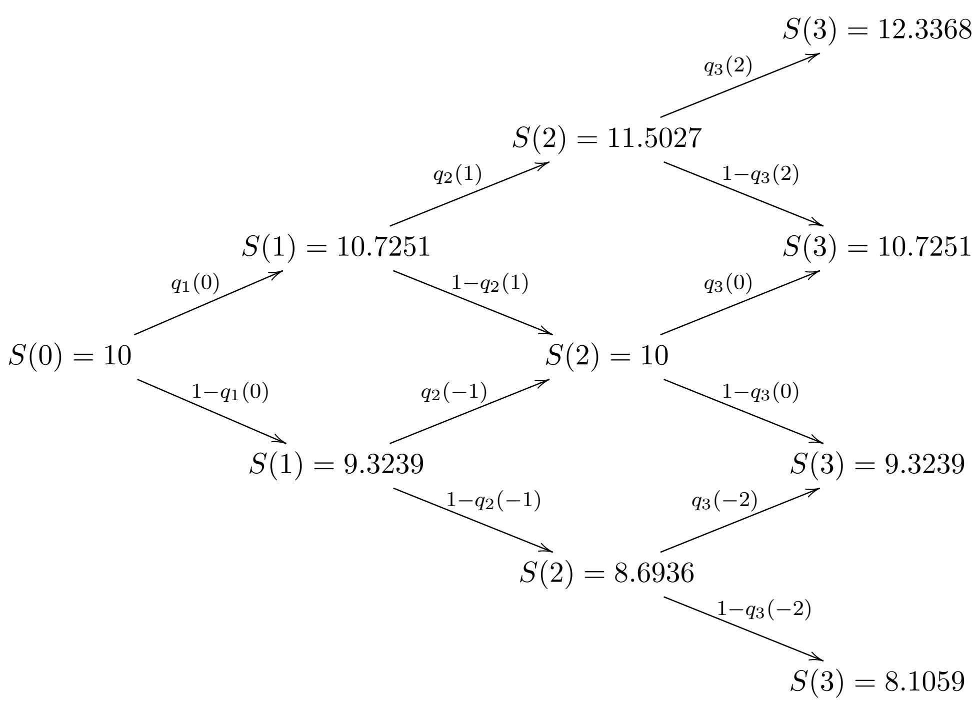
Example in the 3-period model with Ho-Lee risk-free rate. Consider a binomial stock price with $N = 3$, $u = -d = 0.07$, $S_0 = 10$ and a Ho-Lee model for the interest rate with parameters

$$a(0) = R_0 = 0.03, \quad a(1) = 0.05, \quad a(2) = 0.04, \quad b(1) = 0.02, \quad b(2) = 0.01.$$

The martingale transition probabilities are

$$\begin{aligned}
q_1(0) &= \frac{1 + R_0 - e^d}{e^u - e^d} = 0.6966 \\
q_2(1) &= \frac{1 + a(1) + b(1) - e^d}{e^u - e^d} = 0.9821 \\
q_2(-1) &= \frac{1 + a(1) - b(1) - e^d}{e^u - e^d} = 0.6966 \\
q_3(2) &= \frac{1 + a(2) + 2b(2) - e^d}{e^u - e^d} = 0.9107 \\
q_3(0) &= \frac{1 + a(2) - e^d}{e^u - e^d} = 0.7680 \\
q_3(-2) &= \frac{1 + a(2) - 2b(2) - e^d}{e^u - e^d} = 0.6252
\end{aligned}$$

As $q_t(k) \in (0,1)$, the market is complete. The binomial tree for the stock price in the martingale probability is as follows



The binomial tree for the interest rate is

$$
\begin{tikzcd}[column sep=large]
 & & R(2)=0.06 \\
 & R(1)=0.07 \arrow[ur, "q_2(1)"] \arrow[dr, "1-q_2(1)"] & \\
R(0)=0.03 \arrow[ur, "q_1(0)"] \arrow[dr, "1-q_1(0)"'] & & R(2)=0.04 \\
 & R(1)=0.03 \arrow[ur, "q_2(-1)"] \arrow[dr, "1-q_2(-1)"] & \\
 & & R(2)=0.02
\end{tikzcd}
$$

The discount process in the martingale probability has the following distribution

$$D(0)=1, \quad D(1)=\frac{1}{1+R(0)}=0.9709, \quad \text{with prob. } 1,$$

$$D(2)=\frac{D(1)}{1+R(1)}=\begin{cases} \frac{0.9709}{1+0.07}=0.9074, & \text{with prob. } q_1(0) \\ \frac{0.9709}{1+0.03}=0.9426 & \text{with prob. } 1-q_1(0) \end{cases},$$

$$D(3)=\frac{D(2)}{1+R(2)}=\begin{cases} \frac{0.9074}{1+0.06}=0.8560, & \text{with prob. } q_1(0)q_2(1) \\ \frac{0.9074}{1+0.04}=0.8725 & \text{with prob. } q_1(0)(1-q_2(1)) \\ \frac{0.9426}{1+0.04}=0.9063 & \text{with prob. } (1-q_1(0))q_2(-1) \\ \frac{0.9426}{1+0.02}=0.9241 & \text{with prob. } (1-q_1(0))(1-q_2(-1)) \end{cases}$$

Now assume that we want to compute the initial price of a call option on the stock with strike $K=10$ and maturity $T=3$. According to (54), this price is given by

$$\Pi(0)=\widetilde{\mathbb{E}}[D(3)(S(3)-10)_+],$$

where the expectation is in the martingale probability $q_t(k)$. To compute this expectation we need the joint distribution in the risk-neutral probability of the random variables $D(3), S(3)$. Using our results above we find that this joint distribution is given as in the following table:

$D(3)\downarrow$, $S(3)\rightarrow$	12.3368	10.7251	9.3239	8.1059
0.8560	$q_1(0)q_2(1)q_3(2)$	$q_1(0)q_2(1)(1-q_3(2))$	0	0
0.8725	0	$q_1(0)(1-q_2(1))q_3(0)$	$q_1(0)(1-q_2(1))(1-q_3(0))$	0
0.9063	0	$(1-q_1(0))q_2(-1)q_3(0)$	$(1-q_1(0))q_2(-1)(1-q_3(0))$	0
0.9241	0	0	$(1-q_1(0))(1-q_2(-1))q_3(-2)$	$(1-q_1(0))(1-q_2(-1))(1-q_3(-2))$

We conclude that

$$\begin{aligned}\Pi(0)=&\,0.8560[(12.3368-10)q_1(0)q_2(1)q_3(2)+(10.7251-10)q_1(0)q_2(1)(1-q_3(2))] \\ &+0.8725(10.7251-10)q_1(0)(1-q_2(1))q_3(0)+0.9063(1-q_1(0))q_2(-1)q_3(0)=1.4373.\end{aligned}$$

0.5 Probability theory on uncountable sample spaces

In this section we assume that Ω is uncountable (e.g., $\Omega = \mathbb{R}$). In this case there is no general procedure to construct a probability space, but only an abstract definition. In particular a probability measure $\mathbb{P}$ on events $A \subseteq \Omega$ is defined only axiomatically by requiring that $0 \leq \mathbb{P}(A) \leq 1$, $\mathbb{P}(\Omega) = 1$ and that, for any sequence of disjoint events $A_1, A_2, \dots$, it should hold

$$\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$$

Moreover it is not necessary—and almost never convenient—to assume that $\mathbb{P}$ is defined for all events $A \subset \Omega$. We denote by $\mathcal{F}$ the set of events (i.e., subsets of Ω) which have a well defined probability satisfying the properties above.

Example. Let $\Omega = \mathbb{R}$. We say that $A \subseteq \mathbb{R}$ is a **Borel** set if it can be written as the union (or intersection) of countably many open (or closed) intervals. Let $\mathcal{F}$ be the collection of all Borel sets. Let $p : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function such that

$$\int_{\mathbb{R}} p(\omega)\, d\omega = 1.$$

Then $\mathbb{P} : \mathcal{F} \to [0,1]$ given by

$$\mathbb{P}(A) = \int_A p(\omega)\, d\omega \tag{55}$$

defines a probability. If $X : \mathbb{R} \to \mathbb{R}$ is a random variables, the expectation of X in the probability measure (55) is given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} X(\omega) p(\omega)\, dx, \tag{56}$$

provided the integral is finite.

Fortunately for most applications (and in particular for those in financial mathematics) the knowledge of the full probability space is usually not necessary, as in the applications one is typically concerned only with random variables and their distributions, rather than with generic events. More precisely, we are only interested in assigning a probability to events of the form $\{X \in I\}$, where X is a random variable on the (abstract) probability space and $I \subset \mathbb{R}$, that is to say, events which can be resolved by one (or more) random variables.

Remark 0.14. Even though Ω is uncountable, the image of $X : \Omega \to \mathbb{R}$ need not be uncountable (e.g., X could be piecewise constant). To avoid technical complications we assume in the following that $\mathrm{Im}(X)$ does not contain isolated points. We shall refer to these random variables as **continuum random variables**. The only case of non-continuum random variable that we allow in this section is when X is a deterministic constant, in which case the image of X consists of one real number only.

The probability $\mathbb{P}(X \in I)$ can be computed explicitly when X has a density.

Definition 0.3. *Let $f_X : \mathbb{R} \to [0, \infty)$ be a continuous function, except possibly on finitely many points. A continuum random variable $X : \Omega \to \mathbb{R}$ is said to have* **probability density** *f_X if*

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx,$$

for all Borel sets $A \subseteq \mathbb{R}$.

Note that the density f_X satisfies

$$\int_{\mathbb{R}} f_X(x)\, dx = 1$$

and the **cumulative distribution** $F_X(x) = \mathbb{P}(X \le x)$ satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy, \quad \text{for all } x \in \mathbb{R}, \text{ hence } f_X = \frac{dF_X}{dx}.$$

Example. A random variable $X : \Omega \to \mathbb{R}$ is said to be a **normal** random variable with **mean** $m \in \mathbb{R}$ and **variance** $\sigma^2 > 0$ if it admits the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-m|^2}{2\sigma^2}\right). \tag{57}$$

We denote $\mathcal{N}(m, \sigma^2)$ the set of all such random variables. A variable $X \in \mathcal{N}(0,1)$ is called a **standard normal** random variable. The cumulative distribution of standard normal random variables is denoted by $\Phi(x)$ and is called the **standard normal distribution**, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy.$$

The following theorem shows that the probability density, when it exists, provides all the relevant statistical information on a random variable.

Theorem 0.8. *The following holds for all sufficiently regular*[4] *functions $g : \mathbb{R} \to \mathbb{R}$:*

(i) Let $X : \Omega \to \mathbb{R}$ be a random variable with density f_X. Then for all Borel sets $A \subseteq \mathbb{R}$,

$$\mathbb{P}(g(X) \in A) = \int_{x: g(x) \in A} f_X(x)\, dx.$$

(ii) Let $X : \Omega \to \mathbb{R}$ be a random variable with density f_X. Then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(y) f_X(y)\, dy.$$

[4]In particular, for all functions g such that the integrals in the theorem are well-defined.

Moreover the properties 1, 2, 3, 4, 6 in Theorem 0.1 still hold for continuum random variables.

By (ii) in Theorem 0.8, the expectation and the variance of a continuum random variable X with density f_X are given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\,dx, \quad \mathrm{Var}[X] = \int_{\mathbb{R}} x^2 f_X(x)\,dx - \left(\int_{\mathbb{R}} x f_X(x)\,dx\right)^2. \tag{58}$$

Applying (58) to normal variables we obtain

$$X \in \mathcal{N}(m, \sigma^2) \implies \mathbb{E}[X] = m, \quad \mathrm{Var}[X] = \sigma^2. \tag{59}$$

Joint probability density

Definition 0.4. *Two continuum random variables $X, Y : \Omega \to \mathbb{R}$ are said to have the* **joint probability density** $f_{X,Y} : \mathbb{R}^2 \to [0, \infty)$, *if*

$$\mathbb{P}(X \in A, Y \in B) = \int_A \int_B f_{X,Y}(x, y)\,dx\,dy,$$

for all Borel sets $A, B \subseteq \mathbb{R}$.

Note that if $f_{X,Y}$ is a joint probability density, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_{X,Y}(x, y)\,dx\,dy = 1.$$

Moreover if we define the **joint cumulative distribution** as $F_{X,Y}(x, y) = \mathbb{P}(X \le x, Y \le y)$ then

$$f_{X,Y}(x, y) = \partial_x \partial_y F_{X,Y}(x, y).$$

When X, Y have the joint density $f_{X,Y}(x, y)$, the random variables X, Y admit the densities

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)\,dy, \quad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)\,dx.$$

Example: Jointly normally distributed random variables. Let $m \in \mathbb{R}^2$ and $C = (C_{ij})_{i,j=1,2}$ be a symmetric, positive definite 2×2 matrix. Two random variables $X_1, X_2 : \Omega \to \mathbb{R}$ are said to be jointly normally distributed with mean m and **covariance matrix** C if they admit the joint density

$$f_{X_1,X_2}(x) = \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left(-\frac{1}{2}(x - m)C^{-1}(x - m)\right), \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{60}$$

The following theorem generalizes Theorem 0.8 in the presence of two variables.

Theorem 0.9. *Let $X, Y : \Omega \to \mathbb{R}$ be random variables with joint density f_X and $g : \mathbb{R}^2 \to \mathbb{R}$.*

(i) For all Borel sets $A \subseteq \mathbb{R}$ there holds

$$\mathbb{P}(g(X,Y) \in A) = \int_{(x,y):g(x,y)\in A} f_{X,Y}(x,y)\,dx\,dy.$$

(ii) There holds

$$\mathbb{E}[g(X,Y)] = \int_{\mathbb{R}^2} g(x,y) f_{X,Y}(x,y)\,dx\,dy.$$

By (ii) of Theorem 0.9, if X_1, X_2 have the joint density f_{X_1,X_2}, then the covariance of X_1, X_2 can be computed as

$$\begin{aligned}\operatorname{Cov}(X_1,X_2) &= \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] \\ &= \int_{\mathbb{R}^2} x_1x_2 f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 \\ &\quad - \int_{\mathbb{R}^2} x_1 f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 \int_{\mathbb{R}^2} x_2 f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2.\end{aligned}$$

In particular, if X_1, X_2 are jointly normal distributed with mean $m \in \mathbb{R}^2$ and covariance matrix $C = (C_{ij})_{i,j=1,2}$, we find

$$m = (m_1, m_2), \qquad C_{ij} = \operatorname{Cov}(X_i, X_j). \tag{61}$$

The following result on the linear combination of independent normal random variables will play an important role for the project in multi-asset options in Chapter 5.

Theorem 0.10. *Let $X_1, X_2 \in \mathcal{N}(0,1)$ be independent and $a, b, c, d \in \mathbb{R}$. Then $aX_1 + bX_2 \in \mathcal{N}(0, a^2 + b^2)$. Moreover if*

$$Y_1 = aX_1 + bX_2, \quad Y_2 = cX_1 + dX_2,$$

and if the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible, then Y_1, Y_2 are jointly normally distributed with zero mean and covariant matrix $C = AA^T$.

Stochastic processes. Martingales

Let Ω be an uncountable sample space. A stochastic process is a one parameter family $\{X(t)\}_{t\geq 0}$ of (continuum) random variables $X(t) : \Omega \to \mathbb{R}$. We denote $X(t,\omega) = X(t)(\omega)$.

The parameter t is referred to as the time variable, since this is what it represents in the applications that we have in mind. For each $\omega \in \Omega$ *fixed*, the function $t \to X(t, \omega)$ is called a path of the stochastic process. If the paths are all the same for all $\omega \in \Omega$, then we say that $X(t)$ is a deterministic function of time.

Martingale stochastic processes play a fundamental role in options pricing theory[5]. To define martingales on uncountable sample spaces, let $\mathcal{F}_X(t)$ denote the information accumulated by "looking" at the stochastic process up to time t, i.e., the collection of events resolved by $X(s)$ for $0 \leq s \leq t$. Intuitively, the stochastic process $\{X(t)\}_{t\geq 0}$ is a martingale if, based on the information contained in $\mathcal{F}_X(s)$, our "best estimate" on $X(t)$ for $t > s$ is $X(s)$, i.e., we are not able to estimate whether the process will raise or fall in the interval $[s, t]$ with the information available at time s. This intuitive definition is encoded in the formula

$$\mathbb{E}[X(t)|\mathcal{F}_X(s)] = X(s), \quad 0 \leq s \leq t, \tag{62}$$

which generalizes the definition (20) of martingales in finite probability theory. The left hand side of (62) is the conditional expectation of $X(t)$ with respect to the information $\mathcal{F}_X(s)$, whose precise definition is not needed here. It can be shown that (62) implies that martingales have constant expectation.

Brownian motion

Next we recall the definition of the most important of all stochastic processes.

Definition 0.5. *A* **Brownian motion**, *or* **Wiener process**, *is a stochastic process* $\{W(t)\}_{t\geq 0}$ *with the following properties:*

1. *For all*[6] $\omega \in \Omega$, *the paths are continuous (i.e.,* $t \to W(t, \omega)$ *is a continuous function) and* $W(0, \omega) = 0$*;*

2. *For all* $0 = t_0 < t_1 < t_2 < \dots$, *the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1), \dots,$$

are independent random variables;

3. *The increments are normally distributed, that is to say, for all* $0 \leq s < t$,

$$\mathbb{P}(W(t) - W(s) \in A) = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{y^2}{2(t-s)}}\, dy,$$

for all Borel sets $A \subseteq \mathbb{R}$.

[5]In fact, this theory is also called **martingale pricing theory** in some literature.

[6]More precisely, for all $\omega \in \Omega$ up to a set of zero probability.

It can be shown that Brownian motions exist, yet a formal construction is technically quite difficult and beyond the purpose of this text. In particular it can be shown that the process

$$\{W_n(t)\}_{t\in[0,T]}, \quad W_n(t) = \frac{1}{\sqrt{n}} M_{[nt]}, \tag{63}$$

where $[z]$ denotes the greatest integer smaller than or equal to z and $\{M_k\}_{k\in\mathbb{N}}$ is a symmetric random walk, converges (in distribution) to a Brownian motion process. Thus a Brownian motion is the time-continuum limit of a properly rescaled symmetric random walk. In fact, for large $n \in \mathbb{N}$, the process $\{W_n(t)\}_{t\in[0,T]}$ can be used as an approximation for the Brownian motion, which is particularly useful for numerical computations.

The following very simple Matlab function can be used to generate a random path of the Brownian motion:

```
function W=BMPath(T,N)
h=T/N;
W=zeros(1,N);
for j=2:N
W(j)=W(j-1)+sqrt(h)*randn;
end
```

Code 2: Matlab function to simulate a path of the Brownian motion.

Remark 0.15. Since the definition of Brownian motion depends on the probability measure $\mathbb{P}$, then a stochastic process $\{W(t)\}_{t\geq 0}$ which is a Brownian motion in the probability measure $\mathbb{P}$ will in general *not* be a Brownian motion in another probability measure $\widetilde{\mathbb{P}}$. When we want to emphasize that $\{W(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}$, we shall say that $\{W(t)\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion.

Remark 0.16. Letting $s = 0$ in property 3 in Definition 0.5 we obtain that $W(t) \in \mathcal{N}(0,t)$, for all $t > 0$. In particular, $W(t)$ has zero expectation for all times. It can also be shown that Brownian motions are martingales.

The following result is used a few times in the following chapters.

Theorem 0.11. *Let $g : (0,\infty) \to \mathbb{R}$ be a differentiable function and let*

$$X(t) = g(t)W(t) - \int_0^t g'(s)W(s)\,ds.$$

Then

$$X(t) \in \mathcal{N}(0,\Delta(t)), \quad \Delta(t) = \int_0^t g(s)^2\,ds.$$

Remark 0.17. By using the formal identity $d(g(t)W(t)) = g'(t)W(t)dt + g(t)dW(t)$, as well as $\int_0^t d(g(s)W(s)) = g(t)W(t)$, we can write the definition of $X(t)$ in Theorem 0.11 as

$$X(t) = \int_0^t g(s)dW(s),$$

which is called **Itô integral** of the deterministic function $g(t)$, see Section 0.8.

Equivalent probability measures. Girsanov theorem

One further technical complication arising for uncountable sample spaces is the existence of non-trivial events with zero measure, e.g., the event $\{W(t) = 0\}$ that the Brownian motion $W(t)$ takes value zero when $t > 0$. We shall need to consider the concept of equivalent probability measures:

Definition 0.6. *Two probability measure $\mathbb{P}$, $\widetilde{\mathbb{P}}$ on the events $A \in \mathcal{F}$ are said to be equivalent if $\mathbb{P}(A) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$.*

Hence equivalent probability measures agree on which events are impossible. Note that in a finite probability space all probability measures are equivalent, as in the finite case the empty set is the only event with zero probability. The following important theorem characterizes the relation between equivalent probability measures on uncountable sample spaces and is known as the Radon-Nikodým theorem. We denote $\mathbb{I}_A$ the **characteristic function** of the set $A \in \mathcal{F}$, i.e., the random variable taking value $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and zero otherwise.

Theorem 0.12 (Radon-Nikodým theorem)**.** *Let $\mathbb{P} : \mathcal{F} \to [0,1]$ be a probability measure. Then $\widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ is a probability measure equivalent to $\mathbb{P}$ if and only if there exists a random variable $Z : \Omega \to \mathbb{R}$ such that $Z > 0$ (with probability 1), $\mathbb{E}[Z] = 1$ and $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$. Moreover if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent then $\widetilde{\mathbb{E}}[X] = \mathbb{E}[ZX]$, for all random variables $X : \Omega \to \mathbb{R}$.*

For example, assume $\Omega = \mathbb{R}$ and that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are defined as in (55), namely

$$\mathbb{P}(A) = \int_A p(\omega)\, d\omega, \quad \widetilde{\mathbb{P}}(A) = \int_A \widetilde{p}(\omega)\, d\omega,$$

where A is a Borel set and p, $\widetilde{p}$ are two continuous non-negative functions such that

$$\int_{\mathbb{R}} p(\omega)\, d\omega = \int_{\mathbb{R}} \widetilde{p}(\omega)\, d\omega = 1.$$

Then, according to Theorem 0.12 and (56), $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if and only if there exists a function $Z : \mathbb{R} \to \mathbb{R}$ such that $Z > 0$, and

$$\widetilde{\mathbb{P}}(A) = \int_A \widetilde{p}(\omega)\, d\omega = \int_{\mathbb{R}} Z(\omega)\mathbb{I}_A(\omega)p(\omega)\, d\omega = \int_A Z(\omega)p(\omega)\, d\omega.$$

As the equality $\int_A \widetilde{p}(\omega)\,d\omega = \int_A Z(\omega)p(\omega)\,d\omega$ has to be satisfied for all Borel sets $A \subset \mathbb{R}$, then $\widetilde{p}(\omega) = Z(\omega)p(\omega)$ must hold for all $\omega \in \mathbb{R}$ (up to a set with zero probability).

Theorem 0.13 (and Definition). *Let $\{W(t)\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion. Given $\theta \in \mathbb{R}$ and $T > 0$ define*

$$Z_\theta = e^{-\theta W(T) - \frac{1}{2}\theta^2 T}. \tag{64}$$

Then $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$, for all Borel sets $A \subseteq \mathbb{R}$, defines a probability measure equivalent to $\mathbb{P}$, which is called **Girsanov's probability** *with parameter $\theta \in \mathbb{R}$.*

Proof. The proof follows immediately from Theorem 0.12, since the random variable (64) satisfies $Z_\theta > 0$ and

$$\mathbb{E}[Z_\theta] = \mathbb{E}[e^{-\theta W(T) - \frac{1}{2}\theta^2 T}] = \int_{\mathbb{R}} e^{-\theta x - \frac{1}{2}\theta^2 T} \frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\pi T}}\,dx = 1,$$

where we used the density of the normal random variable $W(T) \in \mathcal{N}(0,T)$ to compute the expectation of Z_θ in the probability measure $\mathbb{P}$ (see Theorem 0.8(ii)). $\square$

Note that the Girsanov probability measure $\mathbb{P}_\theta$ depend also on T, but this is not reflected in our notation. In the following we denote by $\mathbb{E}_\theta[\cdot]$ the expectation computed in the probability measure $\mathbb{P}_\theta$ for $\theta \neq 0$. When $\theta = 0$ then $\mathbb{P}_\theta = \mathbb{P}$, in which case the expectation is denoted as usual by $\mathbb{E}[\cdot]$. By Theorem 0.12 we have $\mathbb{E}_\theta[X] = \mathbb{E}[Z_\theta X]$, for all random variables $X : \Omega \to \mathbb{R}$. Moreover we now show that $\mathbb{E}_\theta[W(t)] = -\theta t$. In fact by the Radon-Nikodým theorem we have

$$\mathbb{E}_\theta[W(t)] = \mathbb{E}[Z_\theta W(t)] = \mathbb{E}[e^{-\theta W(T) - \frac{1}{2}\theta^2 T} W(t)].$$

Adding and subtracting $W(t)$ in the exponent of the exponential function we have

$$\mathbb{E}_\theta[W(t)] = \mathbb{E}[e^{-\theta(W(T)-W(t)) - \frac{1}{2}\theta^2 T} e^{-\theta W(t)} W(t)] = \mathbb{E}[e^{-\theta(W(T)-W(t)) - \frac{1}{2}\theta^2 T}]\mathbb{E}[e^{-\theta W(t)} W(t)],$$

where in the last step we used that the random variables $X = e^{-\theta(W(T)-W(t)) - \frac{1}{2}\theta^2 T}$ and $Y = e^{-\theta W(t)} W(t)$ are independent (being functions of the independent random variables $W(T) - W(t)$ and $W(t)$). Using $W(T) - W(t) \in \mathcal{N}(0, T-t)$ and $W(t) \in \mathcal{N}(0,t)$, we can compute the expectations of X and Y as

$$\mathbb{E}[X] = e^{-\frac{1}{2}\theta^2 T} \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} e^{-\theta x - \frac{x^2}{2(T-t)}}\,dx = e^{-\frac{\theta^2}{2}t},$$

$$\mathbb{E}[Y] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\theta x - \frac{x^2}{2t}} x\,dx = -e^{\frac{\theta^2}{2}t}\theta t.$$

Hence $\mathbb{E}_\theta[W(t)] = \mathbb{E}[X]\mathbb{E}[Y] = -\theta t$, as claimed. It follows that $\{W(t)\}_{t\geq 0}$ is *not* a $\mathbb{P}_\theta$-Brownian motion, since Brownian motions, by definition, have zero expectation at any time. Now we can state a fundamental theorem in probability theory with deep applications in financial mathematics, namely Girsanov's theorem[7].

[7]Actually we consider only a special case of this theorem, which suffices for our purposes.

Theorem 0.14. *Let $\{W(t)\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion. Given $\theta \in \mathbb{R}$ and $T > 0$, let $\mathbb{P}_\theta$ be the Girsanov probability measure with parameter θ introduced in Theorem 0.13. Define the stochastic process $\{W^{(\theta)}(t)\}_{t\geq 0}$ by*

$$W^{(\theta)}(t) = W(t) + \theta t. \tag{65}$$

Then $\{W^{(\theta)}(t)\}_{t\geq 0}$ is a $\mathbb{P}_\theta$-Brownian motion.

Note carefully that $\{W^{(\theta)}(t)\}_{t\geq 0}$ is *not* a $\mathbb{P}$-Brownian motion, as it follows by the fact that $\mathbb{E}[W^{(\theta)}(t)] = \theta t$. In particular, according to the probability measure $\mathbb{P}$, the stochastic process $\{W^{(\theta)}(t)\}_{t\geq 0}$ has a *drift*, i.e., a tendency to move up (if $\theta > 0$) or down (if $\theta < 0$). However in the Girsanov probability this drift is removed, because, as shown before, $\mathbb{E}_\theta[W^{(\theta)}(t)] = \mathbb{E}_\theta[W(t)] + \theta t = 0$.

Multi-dimensional Girsanov theorem

We conclude this section with a generalization of Girsanov's theorem in the presence of two independent Brownian motions. This generalization is important for the project on multi-asset options in Chapter 5. We limit ourselves to state without proof the analogs of Theorems 0.13 and 0.14 required for this purpose.

Theorem 0.15 (and Definition). *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be $\mathbb{P}$-independent Brownian motions. Given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $T > 0$ define*

$$Z_\theta = e^{-\theta_1 W_1(T) - \theta_2 W_2(T) - \frac{1}{2}(\theta_1^2 + \theta_2^2)T}. \tag{66}$$

Then $\mathbb{P}_\theta(A) = \mathbb{E}[Z_\theta \mathbb{I}_A]$ defines a probability measure equivalent to $\mathbb{P}$, which is called Girsanov's probability with parameters $\theta_1, \theta_2 \in \mathbb{R}$.

Theorem 0.16. *Let $\{W_1(t)\}_{t\geq 0}$, $\{W_2(t)\}_{t\geq 0}$ be $\mathbb{P}$-independent Brownian motions. Given $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $T > 0$, let $\mathbb{P}_\theta$ be the Girsanov probability with parameters θ_1, θ_2 introduced in Theorem 0.15. Define the stochastic processes $\{W_1^{(\theta)}(t)\}_{t\geq 0}$, $\{W_2^{(\theta)}(t)\}_{t\geq 0}$ by*

$$W_1^{(\theta)}(t) = W_1(t) + \theta_1 t, \quad W_2^{(\theta)}(t) = W_2(t) + \theta_2 t \tag{67}$$

Then $\{W_1^{(\theta)}(t)\}_{t\geq 0}$, $\{W_2^{(\theta)}(t)\}_{t\geq 0}$ are $\mathbb{P}_\theta$-independent Brownian motions.

0.6 Black-Scholes options pricing theory

In the binomial model the stock price at time t is a finite random variable $S(t)$. In the Black-Scholes model the stock price is a continuum random variable with image $\mathrm{Im}(S(t)) = (0, \infty)$, namely the **geometric Brownian motion**

$$S(t) = S_0 e^{\alpha t + \sigma W(t)}. \tag{68}$$

The probability $\mathbb{P}$ with respect to which $\{W(t)\}_{t\geq 0}$ is Brownian motion is the **physical** (or **real-world**) **probability** of the Black-Scholes market. Moreover α is the **instantaneous mean of log-return** and σ^2 is the **instantaneous variance** of the geometric Brownian motion, namely

$$\alpha = \lim_{h\to 0}\frac{1}{h}\mathbb{E}[\widehat{R}_h], \quad \sigma^2 = \lim_{h\to 0}\frac{1}{h}\mathrm{Var}[\widehat{R}_h], \quad \widehat{R}_h = \log S(t+h) - \log S(t). \tag{69}$$

The parameter σ itself is the **instantaneous volatility**.

The geometric Brownian motion admits the density

$$f_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S(0) - \alpha t)^2}{2\sigma^2 t}\right), \tag{70}$$

where $H(x)$ is the **Heaviside** function. It can be shown that the binomial stock price converges in distribution to the geometric Brownian motion in the time-continuum limit, see [2].

The risk-neutral pricing formula in Black-Scholes markets

The purpose of this section is to introduce the definition of Black-Scholes price of European derivatives from a probability theory point of view. Recall that the probabilistic formulation of the binomial options pricing model is encoded in the risk-neutral pricing formula (36). Our goal is to derive a similar risk-neutral pricing formula (at time $t = 0$) for the time-continuum Black-Scholes model.

Motivated by the approach for the binomial model, we first look for a probability measure in which the the discounted stock price in Black-Scholes markets is a martingale (martingale probability measure). It is natural to seek such martingale probability within the class of Girsanov probabilities $\mathbb{P}_\theta$ equivalent to the physical probability $\mathbb{P}$ which we defined in Theorem 0.13. To this purpose we shall need the form of the density function of the geometric Brownian motion in the probability measure $\mathbb{P}_\theta$.

Theorem 0.17. *Let $\theta \in \mathbb{R}$, $T > 0$ and $\mathbb{P}_\theta$ be the Girsanov probability measure equivalent to the physical probability $\mathbb{P}$. The geometric Brownian motion* (68) *has the following density in the probability measure $\mathbb{P}_\theta$:*

$$f^{(\theta)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}}\frac{1}{x}\exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t}\right). \tag{71}$$

Proof. Since

$$S(t) = S_0 e^{\alpha t + \sigma W(t)} = S_0 e^{(\alpha - \theta\sigma)t + \sigma W^{(\theta)}(t)}, \quad W^{(\theta)}(t) = W(t) + \theta t$$

and since $\{W^{(\theta)}(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}_\theta$ (see Girsanov's Theorem 0.14), then the density $f^{(\theta)}_{S(t)}$ is the same as $f_{S(t)}$ with α replaced by $\alpha - \theta\sigma$. $\square$

Let $\mathbb{E}_\theta[\cdot]$ denote the expectation in the probability $\mathbb{P}_\theta$. Recall that martingales have constant expectation. Hence in the martingale (or risk-neutral) probability measure the expectation of the discounted value of the stock must be constant, i.e., $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$. This condition alone suffices to single out a unique possible value of θ.

Theorem 0.18. *The identity $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ holds if and only if $\theta = q$, where*

$$q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2}. \tag{72}$$

Proof. Using the density (71) of $S(t)$ in the measure $\mathbb{P}_\theta$ and (58) we have

$$\mathbb{E}_\theta[S(t)] = \int_{\mathbb{R}} x f^{(\theta)}_{S(t)}(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_0^\infty \exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t}\right) dx.$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha-\theta\sigma)t}{\sigma\sqrt{t}}$, $dx = x\sigma\sqrt{t}\,dy$, we obtain

$$\mathbb{E}_\theta[S(t)] = \frac{S_0}{\sqrt{2\pi}} e^{(\alpha-\theta\sigma)t} \int_{\mathbb{R}} e^{-\frac{y^2}{2} + \sigma\sqrt{t}y}\,dy = S_0 e^{(\alpha-\theta\sigma+\frac{\sigma^2}{2})t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(y+\sigma\sqrt{t})^2}{2}}\,dy.$$

As $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx = 1$, the result follows. □

Even though the validity of $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ is only necessary for the discounted geometric Brownian motion to be a martingale, one can show that the following result holds.

Theorem 0.19. *The discounted value of the geometric Brownian motion stock price is a martingale in the probability measure $\mathbb{P}_\theta$ if and only if $\theta = q$, where q is given by* (72).

The previous discussion leads us to the following definition.

Definition 0.7. *Given $\alpha \in \mathbb{R}$, $\sigma > 0$, $r \in \mathbb{R}$ and $T > 0$, the probability measure*

$$\mathbb{P}_q(A) = \mathbb{E}[e^{-qW(T) - \frac{1}{2}q^2 T}\mathbb{I}_A], \quad q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2}$$

is called the **martingale probability***, or* **risk-neutral probability***, in the interval* $[0, T]$ *of the Black-Scholes market with parameters α, σ, r.*

Remark 0.18. In the risk-neutral probability the stock price is given by the geometric Brownian motion

$$S(t) = S(0) e^{(r - \frac{\sigma^2}{2})t + \sigma\widetilde{W}(t)}, \tag{73}$$

where, by Girsanov's theorem,

$$\widetilde{W}(t) := W^{(q)}(t) = W(t) + \left(\frac{\alpha - r}{\sigma} + \frac{\sigma}{2}\right) t \tag{74}$$

is a Brownian motion in the risk-neutral probability. This follows by replacing $\alpha = r + q\sigma - \frac{1}{2}\sigma^2$ into (68).

At this point we have all we need to define the Black-Scholes price of European derivatives at time $t = 0$ using the risk-neutral pricing formula.

Definition 0.8. *The Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y at maturity T is given by the risk-neutral pricing formula*

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y], \tag{75}$$

i.e., it equals the expected value of the discounted pay-off in the risk-neutral probability measure of the Black-Scholes market.

In the case of standard European derivatives we can use the density of the geometric Brownian motion in the risk-neutral probability measure to write the Black-Scholes price in the following integral form.

Theorem 0.20. *For the standard European derivative with pay-off $Y = g(S(T))$ at maturity $T > 0$, the Black-Scholes price at time $t = 0$ can be written as $\Pi_Y(0) = v_0(S_0)$, where S_0 is the price of the underlying stock at time $t = 0$ and $v_0 : (0, \infty) \to \mathbb{R}$ is the* **pricing function** *of the derivative at time $t = 0$, which is given by*

$$v_0(x) = e^{-rT} \int_{\mathbb{R}} g(xe^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y})e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}. \tag{76}$$

Proof. Replacing $\theta = q$ in (71) we obtain that the geometric Brownian motion has the following density in the risk-neutral probability measure $\mathbb{P}_q$:

$$f^{(q)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right). \tag{77}$$

Using the density (77) for $t = T$ in the risk-neutral pricing formula (75) we obtain

$$\begin{aligned}\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y] = e^{-rT}\mathbb{E}_q[g(S(T))] &= \int_{\mathbb{R}} g(x) f^{(q)}_{S(T)}(x)\, dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 t}} \int_0^\infty \frac{g(x)}{x} \exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right) dx.\end{aligned}$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha - \theta\sigma)t}{\sigma\sqrt{t}}$ we obtain

$$\Pi_Y(0) = e^{-rT} \int_{\mathbb{R}} g(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y})e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} = v_0(S_0),$$

as claimed. $\square$

Remark 0.19. Of course we are tacitly assuming that the pay-off function g is such that the integral in the right hand side of (76) is finite.

For instance, in the case of the European call option with strike K and maturity T, for which the pay-off function is $g(z) = (z - K)_+$, Theorem 0.20 gives

$$\Pi_{\text{call}}(0) = C_0(S_0, K, T), \quad C_0(x, K, T) = x\Phi(d_{(+)}) - Ke^{-rT}\Phi(d_{(-)}) \tag{78a}$$

where Φ is the standard normal distribution and

$$d_{(\pm)} = \frac{\log \frac{x}{K} + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \tag{78b}$$

Definition 0.8 is only valid at time $t = 0$. The risk-neutral pricing formula for $t > 0$ is

$$\Pi_Y(t) = e^{-r(T-t)}\mathbb{E}_q[Y|\mathcal{F}_S(t)], \tag{79}$$

which generalizes (36) to the time continuum case. The right hand side of (79) is the expectation of the discounted pay-off in the risk-neutral probability measure conditional to the information available at time t, which in a Black-Scholes market is determined by the history of the stock price up to time t. It can be shown that in the case of the standard European derivative with pay-off $Y = g(S(T))$ at maturity T, the risk-neutral pricing formula (79) entails that the Black-Scholes price at time $t \in [0, T]$ can be written in the integral form

$$\Pi_Y(t) = v(t, S(t)), \quad \text{where} \quad v(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(xe^{(r-\frac{\sigma^2}{2})\tau}e^{\sigma\sqrt{\tau}\,y}\right) e^{-\frac{y^2}{2}}\,dy, \quad \tau = T - t. \tag{80}$$

Hence the pricing function $v(t, x)$ of the derivative at time t is the same as the pricing function (76) at time $t = 0$ but with maturity T replaced by the time τ left to maturity, which is rather intuitive.

0.7 The Monte Carlo method

The Monte Carlo method is, in its simplest form, a numerical method to compute the expectation of a random variable. Its mathematical validation is based on the **Law of Large Numbers**, which states the following: Suppose $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with expectation $\mathbb{E}[X_i] = \mu$. Then the sample average of the first n components of the sequence, i.e.,

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n),$$

converges (in probability) to μ as $n \to \infty$.

The law of large numbers can be used to justify the fact that if we are given a large number of independent trials $X_1, \ldots, X_n$ of a random variable X, then

$$\mathbb{E}[X] \approx \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

To measure how reliable is the approximation of $\mathbb{E}[X]$ given by the sample average, consider the standard deviation of the trials $X_1, \dots, X_n$:

$$s_X = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\overline{X} - X_i)^2}.$$

Viewing $X_1, \dots, X_n$ as independent copies of X, a simple application of the Central Limit Theorem proves that the random variable

$$\frac{\mu - \overline{X}}{s_X/\sqrt{n}}$$

converges in distribution to a standard normal random variable. We use this result to show that the true value μ of $\mathbb{E}[X]$ has about 95% probability to be in the interval

$$[\overline{X} - 1.96\frac{s}{\sqrt{n}}, \overline{X} + 1.96\frac{s}{\sqrt{n}}].$$

Indeed, for n large,

$$\mathbb{P}\left(-1.96 \leq \frac{\mu - \overline{X}}{s_X/\sqrt{n}} \leq 1.96\right) \approx \int_{-1.96}^{1.96} e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}} \approx 0.95.$$

An application to Black-Scholes theory

Using the Monte Carlo method and the risk-neutral pricing formula (37), we can approximate the Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y and maturity $T > 0$ with the sample average

$$\Pi_Y(0) = e^{-rT}\frac{Y_1 + \dots Y_n}{n}, \tag{81}$$

where $Y_1, \dots, Y_n$ is a large number of independent trials of the pay-off. Each trial Y_i is determined by a path of the stock price. Letting $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$, we may construct a sample of n paths of the geometric Brownian motion on the given partition with the following simple Matlab function:

Note carefully that the stock price is modeled as a geometric Brownian motion with mean of log return $\alpha = r - \sigma^2/2$, which means that the geometric Brownian motion is risk-neutral, see (77). This is of course correct, since the expectation in (81) that we want to compute is in the risk-neutral probability measure. The following Matlab code compute the Black-Scholes price of a call option using the Monte Carlo method. The code also computes the statistical error

$$\mathrm{Err} = 1.96\frac{s}{\sqrt{n}} \tag{82}$$

```
function S=GBMPaths(s,sigma,r,T,N,n)
S=zeros(n,N);
t=linspace(0,T,N);
for i=1:n
W=BMPath(T,N);
S(i,:)=s*exp((r-sigma^2/2)*t(:)+sigma*W(:));
end
```

Code 3: Matlab function to simulate a n paths of the geometric Brownian motion.

```
function [price, conf95]=MonteCarloCall(s,sigma,r,K,T,N,n)
tic
S=GBMPaths(s,sigma,r,T,N,n);
payOff=max(0,S(:,N)-K);
price=exp(-r*T)*mean(payOff);
conf95=1.96*std(payOff)/sqrt(n);
toc
```

Code 4: Matlab function to compute the initial Black-Scholes price of European calls using the Monte Carlo method.

of the Monte Carlo price, where s is the standard deviation of the pay-off trials.

For instance, by running the command

$$[\texttt{price}, \texttt{conf95}] = \texttt{MonteCarloCall}(\texttt{10}, \texttt{0.5}, \texttt{0.01}, \texttt{10}, \texttt{1}, \texttt{100}, \texttt{100000})$$

we obtain the output

```
price = 1.9976
conf95 = 0.0249
```

The calculation took about half a second. The exact price for the given call obtained by using the formula (78) is 2.0144, which lies within the confidence interval $[1.9976-0.0249, 1.9976+0.0249] = [1.9727, 2.0225]$ of the Monte Carlo price. Remark: The formula (78) is implemented in Matlab by the function `blsprice`.

Control variate Monte Carlo

The Monte Carlo method just described is also known as **crude** Monte Carlo and can be improved in a number of ways. For instance, it follows by (82) that in order to reduce the error of the Monte Carlo price, one needs to either (i) increase the number of trials n or (ii) reduce the standard derivation s. As increasing n can be very costly in terms of computational time, the approach (ii) is preferable. There exist several methods to decrease the standard deviation of a Monte Carlo computation, which are collectively called **variance reduction techniques**. Here we describe the **control variate** method.

Suppose we want to compute $\mathbb{E}[X]$. The idea of the control variate method is to introduce a second random variable Q for which $\mathbb{E}[Q]$ can be computed *exactly* and then write

$$\mathbb{E}[X] = \mathbb{E}[Y] + \mathbb{E}[Q], \quad \text{where } Y = X - Q.$$

Hence the Monte Carlo approximation of $\mathbb{E}[X]$ can now be written as

$$\mathbb{E}[X] \approx \frac{Y_1 + \cdots + Y_n}{n} + \mathbb{E}[Q],$$

where $Y_1, \ldots, Y_n$ are independent trials of the random variable Y. This approximation improves the crude Monte Carlo estimate (i.e., without control variate) if the sample average estimator of $\mathbb{E}[Y]$ is better than the sample average estimator of $\mathbb{E}[X]$. Because of (82), this will be the case if $(s_Y)^2 < (s_X)^2$. It will now be shown that the latter inequality holds if X, Q *have a positive large correlation*. Letting $X_1, \ldots, X_n$ be independent trials of X and $Q_1, \ldots, Q_n$ be independent trials of Q, we compute

$$\begin{aligned}(s_Y)^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(\overline{Y} - Y_i)^2 = \frac{1}{n-1}\sum_{i=1}^{n}((\overline{X} - \overline{Q}) - (X_i - Q_i))^2 \\ &= (s_X)^2 + (s_Q)^2 - 2C(X, Q),\end{aligned}$$

where $C(X, Q)$ is the sample covariance of the trials $(X_1, \ldots, X_n)$, $(Q_1, \ldots, Q_n)$, namely

$$C(X, Q) = \sum_{i=1}^{n}(\overline{X} - X_i)(\overline{Q} - Q_i).$$

Hence $(s_Y)^2 < (s_X)^2$ holds provided $C(X, Q)$ is sufficiently large and positive (precisely, $C(X, Q) > s_Q/\sqrt{2}$). As $C(X, Q)$ is an unbiased estimator of $\mathrm{Cov}(X, Q)$, then the use of the control variate Q will improve the performance of the crude Monte Carlo method if X, Q have a positive large correlation. An application of this method to the Asian option is one of the goals of the project in Chapter 3.

0.8 Introduction to Itô's integral and stochastic calculus

In Black-Scholes theory the price of the stock is a deterministic function of the Brownian motion, namely, $S(t) = f(t, W(t))$, where $f(t, x) = S_0 \exp(\alpha t + \sigma x)$, see (68). In this section we discuss more general models in which the stock price at time t depends on the whole path of the Brownian motion in the interval $[0, t]$ and not just on $W(t)$. In these models the

Brownian motion still remains the only source of randomness, but we have now access to a much larger variety of market models.

The simplest generalization consists in assuming that the stock price $S(t)$ depends on the time integral of the Brownian motion in the interval $[0,t]$, as in the following example

$$S(t) = S_0 \exp\left(\int_0^t \alpha(\tau)\, d\tau - \int_0^t \sigma'(\tau) W(\tau) d\tau + \sigma(t) W(t)\right), \tag{83}$$

where $\alpha(t), \sigma(t)$ are continuously differentiable (deterministic) functions of time. Note that when α and σ are constant, (83) reduces to the geometric Brownian motion (68).

Exercise 0.1. *Prove that $\alpha(t)$ is the instantaneous mean of log-return and $\sigma(t)^2$ is the instantaneous variance of the stock price* (83)*.*

The stock price (83) contains the time integral of the stochastic process $X(t) = \sigma'(t)W(t)$. Since $\sigma(t)$ is continuously differentiable and the Brownian motion has continuous paths, then $t \to X(t)$ is continuous and therefore the integral in (83) can be understood in the standard Riemann sense. We recall that the Riemann integral of a continuous function $g : [0,t] \to \mathbb{R}$ in the interval $[0,t]$ is defined as the limit of the Riemann sum, namely

$$\int_0^t g(\tau)\, d\tau = \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} g(t_i^{(n)})(t_{i+1}^{(n)} - t_i^{(n)}), \tag{84}$$

where $\pi_n = \{0 = t_0^{(n)}, t_1^{(n)}, \dots, t_{m(n)}^{(n)} = t\}$ is an increasing sequence partitions of the interval $[0,t]$ (i.e., $\pi_n \subset \pi_{n+1}$) such that

$$\max_{i=0,\dots,m(n)-1} (t_{i+1}^{(n)} - t_i^{(n)}) \to 0 \quad \text{as } n \to \infty$$

(i.e., the length of each subinterval of the partition tends to zero in the limit $n \to \infty$).

Another way to introduce an integral stochastic process from the Brownian motion is to replace the time increments in (84) with the increments of the Brownian motion. More precisely, given a stochastic process $\{X(t)\}_{t\geq 0}$ with continuous paths and measurable with respect to the Brownian motion, define

$$I_n(t) = \sum_{i=0}^{m(n)-1} X(t_i^{(n)})(W(t_{i+1}^{(n)}) - W(t_i^{(n)})).$$

Note that the random variables $X(t_i^{(n)})$ and $W(t_{i+1}^{(n)}) - W(t_i^{(n)})$ within the sum are independent, because $X(t_i^{(n)})$ depends only on the Brownian motion up to time $t_i^{(n)}$ (as $\{X(t)\}_{t\geq 0}$ is assumed to be measurable with respect to $\{W(t)\}_{t\geq 0}$), and the increment $W(t_{i+1}^{(n)}) - W(t_i^{(n)})$ is independent of $W(t_i^{(n)})$ (by definition of Brownian motion). It follows that

$$\mathbb{E}[I_n(t)] = \sum_{i=0}^{m(n)-1} \mathbb{E}[X(t_i^{(n)})(W(t_{i+1}^{(n)}) - W(t_i^{(n)}))] = \sum_{i=0}^{m(n)-1} \mathbb{E}[X(t_i^{(n)})]\mathbb{E}[W(t_{i+1}^{(n)}) - W(t_i^{(n)})] = 0. \tag{85}$$

The Itô's integral of $X(t)$ in the interval $[0,t]$ is defined as the limit (in probability) of the random variable $I_n(t)$ when $n \to \infty$ and is denoted as follows:

$$\int_0^t X(\tau) dW(\tau) = \lim_{n\to\infty} I_n(t).$$

By (85), the Itô's integral has zero expectation for all times:

$$\mathbb{E}[\int_0^t X(\tau) dW(\tau)] = 0, \quad \text{for all } t > 0. \tag{86}$$

Note carefully that Itô's integrals are random variables, and thus $\{I(t)\}_{t\geq 0}$ is a stochastic process. In fact, under mild regularity assumptions on the integrand stochastic process $\{X(t)\}_{t\geq 0}$, $\{I(t)\}_{t\geq 0}$ *is a martingale.* $I(t)$ can be expressed in terms of the Brownian motion and Riemann integrals of the Brownian motion by using the following fundamental result.

Theorem 0.21 (Itô's formula)**.** *Let $f = f(t,x)$ be a function with continuous partial derivatives $\partial_t f, \partial_x f, \partial_x^2 f$. Then*

$$f(t,W(t)) = f(0,0) + \int_0^t \left[\partial_t f(\tau,W(\tau)) + \frac{1}{2}\partial_x^2 f(\tau,W(\tau))\right] d\tau + \int_0^t \partial_x f(\tau,W(\tau))\, dW(\tau). \tag{87}$$

Examples

- Choosing $f(t,x) = f(x) = x^2$, Itô's formula becomes

$$W(t)^2 = t + 2\int_0^t W(\tau) dW(\tau), \quad \text{hence} \quad \int_0^t W(\tau)\, dW(\tau) = \frac{1}{2}W(t)^2 - \frac{t}{2}.$$

- Choosing $f(t,x) = \sigma(t)x$, where σ is a continuously differentiable (deterministic) function of time, we obtain

$$\sigma(t)W(t) = \int_0^t \sigma'(\tau)W(\tau)\, d\tau + \int_0^t \sigma(\tau)\, dW(\tau),$$

  hence we can rewrite the stochastic process (83) as

$$S(t) = S_0 \exp\left(\int_0^t \alpha(\tau)\, d\tau + \int_0^t \sigma(\tau)\, dW(\tau)\right).$$

- Choosing $f(t,x) = tx$ we obtain

$$\int_0^t \tau\, dW(\tau) = tW(t) - \int_0^t W(\tau)\, d\tau$$

  and thus we express the Itô integral on the left in terms of $W(t)$ and of the standard Riemann integral of $W(t)$ on the right.

Exercise 0.2. *Use Itô's formula to express the following Itô integrals in terms of $W(t)$ and Riemann integrals of $W(t)$:*

$$\int_0^t W(\tau)^n \, dW(\tau) \ (n>0), \quad \int_0^t \cos(\tau\, W(\tau))\, dW(\tau), \quad \int_0^t \frac{dW(\tau)}{\sqrt{1+W(\tau)^2}}.$$

Itô processes and stochastic differential equations

Now that we have at our disposal two different notions of integral of stochastic processes, we can combine them to construct a general class of stochastic processes.

Definition 0.9. *Let $\{A(t)\}_{t\geq 0}$ and $\{B(t)\}_{t\geq 0}$ be stochastic processes with continuous paths and measurable with respect to the Brownian motion $\{W(t)\}_{t\geq 0}$. The stochastic process*

$$X(t) = X(0) + \int_0^t A(\tau)\, d\tau + \int_0^t B(\tau)\, dW(\tau) \tag{88}$$

is called Itô's process with **drift rate** *$A(t)$ and* **diffusion rate** *$B(t)$. If $A(t), B(t)$ are deterministic functions of $X(t)$ itself, that is if*

$A(t) = a(t, X(t)), \quad B(t) = b(t, X(t)),$ *for some (smooth) functions $a, b : (0,\infty)\times\mathbb{R}\to\mathbb{R}$*

then (88) *is called a* **stochastic differential equation** *(or SDE).*

As we shall see later, stochastic processes in finance are SDE's. When $A\equiv 0$ the Itô process reduces to a pure Itô integral (plus the time independent random variable $X(0)$) and thus it is a martingale. .

Itô's formula in Theorem 0.21 generalizes to Itô's processes as follows.

Theorem 0.22 (Itô's formula)**.** *Let $f = f(t,x)$ be a function with continuous partial derivatives $\partial_t f, \partial_x f, \partial_x^2 f$ and $\{X(t)\}_{t\geq 0}$ the Itô process* (88) *Then*

$$\begin{aligned} f(t, X(t)) =& f(0, X(0)) + \int_0^t \left(\partial_t f(\tau, X(\tau)) + A(\tau)\partial_x f(\tau, X(\tau)) + \frac{1}{2}B(\tau)^2\partial_x^2 f(\tau, X(\tau))\right) d\tau \\ &+ \int_0^t B(\tau)\partial_x f(\tau, X(\tau))\, dW(\tau). \end{aligned} \tag{89}$$

By the previous theorem, if $\{X(t)\}_{t\geq 0}$ is an Itô process with drift rate $\{A(t)\}_{t\geq 0}$ and diffusion rate $\{B(t)\}_{t\geq 0}$, then $\{f(t, X(t))\}_{t\geq 0}$ is an Itô process with drift rate $\{A_*(t)\}_{t\geq 0}$ and diffusion rate $\{B_*(t)\}_{t\geq 0}$ given by

$$A_*(t) = \partial_t f(t, X(t)) + A(t)\partial_x f(t, X(t)) + \frac{1}{2}B(t)^2\partial_x^2 f(t, X(t)), \quad B_*(t) = B(t)\partial_x f(t, X(t)).$$

In the particular case when $\{X(t)\}_{t\geq 0}$ is a SDE, the drift rate $\{A_*(t)\}_{t\geq 0}$ of the process $\{f(t,X(t))\}_{t\geq 0}$ is given by

$$A_*(t) = \left[\partial_t f(t,x) + a(t,x)\partial_x f(t,x) + \frac{1}{2}b(t,x)^2\partial_x^2 f(t,x)\right]_{x=X(t)}.$$

Hence we obtain the following corollary of Theorem 0.22.

Corollary 0.1. *Let $\{X(t)\}_{t\geq 0}$ be the stochastic differential equation*

$$X(t) = X(0) + \int_0^t a(\tau, X(\tau))\,d\tau + \int_0^t b(\tau, X(\tau))\,dW(\tau). \tag{90}$$

If $f(t,x)$ satisfies the partial differential equation (PDE)

$$\partial_t f(t,x) + a(t,x)\partial_x f(t,x) + \frac{1}{2}b(t,x)^2\partial_x^2 f(t,x) = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{91}$$

then the stochastic process $\{f(t,X(t))\}_{t\geq 0}$ is a martingale.

Equation (91) is called the **(backward) Kolmogorov PDE** associated to the SDE (90). Before discussing some examples, it is convenient to introduce the so-called **stochastic differential notation**. In this notation, the Itô process (88) is written as

$$dX(t) = A(t)\,dt + B(t)\,dW(t).$$

Note that the d in $dX(t)$ and $dW(t)$ is *not* a differential operator, because $X(t)$ and $W(t)$ are *not* differentiable functions. However if we formally integrate the above expression from 0 to t and use $\int_0^t dX(\tau) = X(t) - X(0)$, then we go back to the original integral notation in (88). Itô's formula in Theorem 0.22 reads, in this notation,

$$df(t,X(t)) = [\partial_t f(t,X(t)) + A(t)\partial_x f(t,X(t)) + \frac{1}{2}B(t)^2\partial_x^2 f(t,X(t))]\,dt + B(t)\partial_x f(t,X(t))\,dW(t). \tag{92}$$

Moreover the general SDE can be written as

$$dX(t) = a(t,X(t))\,dt + b(t,X(t))\,dW(t). \tag{93}$$

If $b \equiv 0$ in the previous expressing and we "divide" by dt, the SDE becomes the ordinary differential equation (ODE) $\frac{dX(t)}{dt} = a(t,X(t))$. In fact, SDE's can be interpreted as generalizations of ODE's in which we add a random term in the right hand side.

Remark 0.20. In this short introduction to stochastic calculus we use the stochastic differential notation only as a way to simplify our formulas, but it is actually a very powerful tool, see [2].

SDE's in finance

As mentioned before, all stochastic processes in finance are given in the form of stochastic differential equations. Let us show for instance that the geometric Brownian motion (68) is an SDE. First we write (68) in the form

$$S(t) = S_0 e^{X(t)} = f(X(t)), \quad X(t) = \alpha t + \sigma W(t), \quad f(x) = S_0 e^x. \tag{94}$$

Hence by Itô's formula in Theorem 0.22 we find (using the stochastic differential notation (92))

$$dS(t) = (\alpha S_0 e^{X(t)} + \frac{1}{2}\sigma^2 S_0 e^{X(t)})\, dt + \sigma S_0 e^{X(t)} dW(t),$$

i.e.,

$$dS(t) = (\alpha + \frac{1}{2}\sigma^2) S(t)\, dt + \sigma S(t)\, dW(t). \tag{95}$$

Thus the geometric Brownian motion has the form (100) with $a(t,x) = (\alpha + \frac{1}{2}\sigma^2)x$ and $b(t,x) = \sigma x$. In particular, the geometric Brownian motion is a **linear stochastic differential equation**, since the functions a, b are linear in the x-variable. In this context, the geometric Brownian motion expressed in the form (94), i.e., as a deterministic function of $W(t)$, is said to be the solution of the SDE (95) (with initial datum $S(0) = S_0$).

The SDE (95) is posed in the physical probability, because it contains the Brownian motion $W(t)$. If we replace $W(t)$ with $\widetilde{W}(t)$ given by (74), i.e.,

$$dW(t) = d\widetilde{W}(t) - \left(\frac{\alpha - r}{\sigma} + \frac{\sigma}{2}\right) dt,$$

the SDE (95) becomes

$$dS(t) = rS(t)\, dt + \sigma S(t)\, d\widetilde{W}(t), \tag{96}$$

which can also be obtained by directly applying Itô's formula to (73). The SDE (96) implies at once that the discounted stock price $\{e^{-rt}S(t)\}_{t\geq 0}$ is a martingale in the risk-neutral probability. In fact, by Itô's formula (92) with $f(t,x) = e^{-rt}x$, we have

$$d(e^{-rt}S(t)) = \sigma S(t) e^{-rt} d\widetilde{W}(t),$$

that is, $\{e^{-rt}S(t)\}_{t\geq 0}$ is a SDE with zero drift and therefore it is a martingale in the risk-neutral probability.

Stochastic differential equations can be used to generalize the Black-Scholes model in different ways. For instance, in the so-called **constant elasticity variance** model, the stock price in the risk-neutral probability is given by the following SDE:

$$dS(t) = rS(t)\, dt + \sigma S(t)^\gamma\, d\widetilde{W}(t), \tag{97}$$

where γ is a positive constant (the Black-Scholes model is recovered for $\gamma = 1$). The CEV model is useful to reproduce skew implied volatility curves.

SDE's are also used in finance to model stochastic risk-free rates and stochastic volatilities. For instance, given positive constants a, b, c, the SDE on the risk-free rate

$$dr(t) = a(b - r(t))\,dt + c\sqrt{r(t)}\,dW(t) \tag{98a}$$

is called CIR (Cox-Ingersoll-Ross) model, while the same SDE applied to the instantaneous variance $\nu(t) = \sigma(t)^2$, i.e.,

$$d\nu(t) = a(b - \nu(t))\,dt + c\sqrt{\nu(t)}\,dW(t) \tag{98b}$$

is called Heston model. Note that the SDE's (97)-(98) are non-linear and thus cannot in general be solved explicitly (as we did above for the geometric Brownian motion; see also the next exercise). However they can easily be solved numerically, as shown in the following section.

Exercise 0.3. *Consider the SDE*

$$dX(t) = \mu(\theta - X(t))\,dt + c\,dW(t), \tag{99}$$

where μ, θ, c are constants. Solve (99) *(i.e., find $X(t)$ as a function of $W(t)$ and Riemann integrals of $W(t)$) and show that $X(t)$ is normally distributed. Find $\mathbb{E}[X(t)]$, $\mathrm{Var}[X(t)]$. Finally, find functions $\alpha(t), \beta(t)$ such that the process*

$$Y(t) = e^{-\alpha(t)X(t)-\beta(t)}$$

is a martingale. HINTS: To solve the SDE, apply Itô's formula to $e^{\mu t}X(t)$, then apply Theorem 0.11 to the resulting expression of $X(t)$ to prove that it is normally distributed. For the last part of the exercise, find $\alpha(t), \beta(t)$ for which the drift of $Y(t)$ is zero (see Corollary (0.1)*).*

Numerical solutions of SDE's

The problem under discussion in this section is the following: Given a SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t) \tag{100}$$

how can we compute numerically a generic path of the stochastic process $\{X(t)\}_{t\geq 0}$? If the SDE can be solved explicitly in terms of $W(t)$ and Riemann integrals of $W(t)$, then all we need to do is to construct paths of the Brownian motion, for instance using Code 2. This works in particular for the geometric Brownian motion, i.e., within the Black-Scholes model. However for more complex market models, where the stock price or other parameters are non-linear SDE's (as in the examples discussed in the previous section), the paths of $\{X(t)\}_{t\geq 0}$ must be simulated by using directly the SDE (100). The simplest way to do this is by applying the so-called Euler-Maruyama method, which is the generalization to SDE's of the forward Euler method for ODE's.

Consider the SDE (100) with **initial datum** $X(0) = X_0$, which we assume to be a constant (e.g., the initial stock price). Given the uniform partition

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_j = j\frac{T}{N}, \ \Delta t = t_{j+1} - t_j = \frac{T}{N}$$

of the interval $[0, T]$, we define

$$X(t_j) = X_j, \quad j = 0, \ldots, N, \quad W_j = W(t_j).$$

Note that X_j, W_j are random variables and that

$$G_j = \frac{W_j - W_{j-1}}{\sqrt{\Delta t}}$$

are independent standard normal random variables for $j = 1, \ldots, N$. The (explicit) finite difference approximation of (100) is

$$X_j = X_{j-1} + a(t_{j-1}, X_{j-1})\Delta t + b(t_{j-1}, X_{j-1})\sqrt{\Delta t}\, G_j. \tag{101}$$

The following Matlab function applies the iterative equation (101) to the geometric Brownian motion SDE (96), for which (101) becomes

$$S_j = S_{j-1} + rS_{j-1}\Delta t + \sigma S_{j-1}\sqrt{\Delta t}\, G_j, \quad S(0) = S_0,$$

where $S_0 > 0$ is the initial stock price. The output `S` contains one path of the stochastic process $S(t)$ along the time partition $\{t_0 = 0, t_1, \ldots, t_N = T\}$, that is, a path $(S(0) = S_0, S(t_1), \ldots, S(t_N) = S(T))$.

```
function S = GbmSDE(s0,r,sigma,T,N)
dt=T/N;
S=zeros(1,N);
t=zeros(1,N);
G=randn(1,N);
S(1)=s0;
for j=2:N
S(j)=S(j-1)+r*S(j-1)*dt+sigma*S(j-1)*sqrt(dt)*G(j-1);
t(j)=t(j-1)+dt;
end
```

Code 5: Matlab function to simulate a path of the geometric Brownian motion by solving numerically the SDE (96).

Exercise 0.4 (Matlab)**.** *Write a Matlab function that generates n random paths of the*

SDE (97) for the stock price in the CEV model using the Euler-Maruyama method. Then apply the Monte Carlo method to compute the price of call/put options at time $t = 0$ in the CEV model and compare the results obtained with the Black-Scholes price (use $\gamma \in [0.8, 1.2]$ with 10^5 paths). Plot (in the same figure) the option price at $t = 0$ as a fuction of $S(0)$ for different values γ and (in another figure) the option price as function of γ. Discuss your findings. To measure the accuracy of your results, check that the option prices in the CEV model and the Black-Scholes model are the same for $\gamma = 1$ and verify the put-call parity for different values of γ.

Chapter 1

A project on the trinomial model

As opposed to the binomial options pricing model, the trinomial model is an **incomplete** model, that is to say, the risk-neutral price of European derivatives in a trinomial market is not uniquely defined by the arbitrage-free pricnciple. Some scholars believe that real markets are incomplete, because investors assign different values to the market price of risk (i.e., choose a different risk-neutral probability to price European derivatives). In this project the trinomial model is studied in details, in particular regarding the problem of pricing and hedging European derivatives by "almost" self-financing and hedging portfolios.

1.1 The trinomial model

In the trinomial model the stock price is allowed to move in three different directions at each time step, namely $S(0) = S_0 > 0$ and

$$S(t) = \begin{cases} S(t-1)e^u & \text{with prob. } p_u \\ S(t-1)e^m & \text{with prob. } p_m \\ S(t-1)e^d & \text{with prob. } p_d \end{cases} \qquad t = 1, \dots, N,$$

where $u > m > d$, $0 < p_u, p_m, p_d < 1$ and $p_u + p_m + p_d = 1$. The risk-free asset has value $B(t) = B_0 e^{rt}$, where r is constant.

The possible prices of the stock at time $t = 0, \dots, N$ satisfy

$$S(t) \in \{S_0 e^{N_u(t)u + N_d(t)d + (t - N_u(t) - N_d(t))m} \ \text{ for } N_u(t), N_d(t) = 0, \dots, t \text{ and } N_u(t) + N_d(t) \leq t\}.$$

It follows that the number of possible stock prices at time t is

$$\sum_{N_u=0}^{t} \sum_{N_d=0}^{t-N_u} 1 = \sum_{N_u=0}^{t} (t - N_u + 1) = (t+1)t + t + 1 - \sum_{N_u=0}^{t} N_u$$
$$= (t+1)t + t + 1 - \frac{(t+1)t}{2} = \frac{(t+1)(t+2)}{2}.$$

Thus the number of nodes in the trinomial tree grows quadratically—while we recall that for the binomial model this grow was linear ($t+1$ possible prices at time t). To reduce the number of nodes in the trinomial tree we shall assume from now on that the recombination condition holds:

$$m = \frac{u+d}{2}$$

and thus restrict the trinomial stock price to the form

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{with prob. } p_u \\ S(t-1)e^{\frac{u+d}{2}} & \text{with prob. } p_m \\ S(t-1)e^{d} & \text{with prob. } p_u \end{cases} \qquad t = 1, \dots, N, \tag{1.1}$$

with $u > d$. In this case the possible stock prices at time t belong to the set

$$\{S_0 e^{(u-d)(N_u(t)-N_d(t))/2+(u+d)t/2)}, N_u(t), N_d(t) = 0, \dots, t\},$$

which contains $2t+1$ elements. Hence the number of nodes of the trinomial tree with the recombination condition grows linearly, as for the binomial model. In the following we restrict to this case for simplicity.

Probabilistic formulation. Let $\Omega = \{-1, 0, 1\}^N$. Given $p = (p_u, p_m, p_d)$ such that $0 < p_u, p_m, p_d < 1$ and $p_u + p_m + p_d = 1$, we define the probability $\mathbb{P}_p$ on the sample space Ω by letting

$$\mathbb{P}_p(\omega) = p_u^{N_+(\omega)} p_m^{N_0(\omega)} p_d^{N_-(\omega)},$$

where $N_\pm(\omega)$ is the number of ± 1 in the sample ω and $N_0(\omega) = N - N_+(\omega) - N_-(\omega)$ the number of 0's. The trinomial stock price can be regarded as a stochastic process in the probability space $(\Omega, \mathbb{P}_p)$. To see this let the stochastic process $\{X_t\}_{t=1,\dots,N}$ be defined on $\omega = (\gamma_1, \dots, \gamma_N) \in \Omega$ as $X(\omega) = \gamma_t$, that is

$$X_t(\omega) = \begin{cases} -1 & \text{if } \gamma_t = -1 \\ 0 & \text{if } \gamma_t = 0 \\ 1 & \text{if } \gamma_t = 1 \end{cases} . \tag{1.2}$$

Note that the random variables $X_1, \dots, X_N$ are independent and identically distributed (**i.i.d.**). We can write (1.1) as

$$S(t) = S(t-1) \exp\left[\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) X_t\right]. \tag{1.3}$$

Iterating the previous identity, the trinomial stock price at time $t = 1, \dots, N$ is

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) Z_t\right], \quad Z_t = X_1 + \cdots + X_t. \tag{1.4}$$

Hence $S(t) : \Omega \to \mathbb{R}$ and $\{S(t)\}_{t=0,\dots,N}$ is a stochastic process on the probability space $(\Omega, \mathbb{P}_p)$. Letting $Z_0 = 0$, the process $\{S(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{Z_t\}_{t=0,\dots,N}$. Moreover we have the following analogue of Theorem 0.4.

Theorem 1.1. *The probability measure $\mathbb{P}_p$ is a martingale measure if and only if $p = q = (q_u, q_m, q_d)$, where (q_u, q_m, q_d) satisfy*

$$q_u e^u + q_m e^{\frac{u+d}{2}} + q_d e^d = e^r, \tag{1.5a}$$
$$q_u + q_m + q_d = 1, \tag{1.5b}$$
$$0 < q_u, q_m, q_d < 1. \tag{1.5c}$$

Task 1.1. *Prove the theorem.*

We remark that there exists infinitely many triples that satisfy (1.5). Indeed the solution of (1.5a)-(1.5b) can be written in parametric form as

$$q_u = \frac{e^r - e^d}{e^u - e^d} - \omega \frac{e^{d/2}}{e^{u/2} + e^{d/2}}, \quad q_m = \omega, \quad q_d = \frac{e^u - e^r}{e^u - e^d} - \omega \frac{e^{u/2}}{e^{u/2} + e^{d/2}} \tag{1.6}$$

and, under suitable conditions on the market parameters r, u, d and the free parameter ω, all such solutions define a probability, i.e., they satisfy (1.5c). Note also that in the limit $\omega \to 0$ the trinomial model reduces to the binomial model and the solutions (1.6) converge to the martingale probability measure of the binomial model, see Theorem 0.4.

Task 1.2. *Let $r > 0$, $u > 0$ and $u = -d$. Show that the triples* (1.6) *satisfy* (1.5c) *if and only if*

$$u > r \quad \text{and} \quad 0 < \omega < \frac{e^u - e^r}{e^u - 1}.$$

The existence of a martingale probability measure ensures that the trinomial market is free of self-financing arbitrages, see Remark 0.6. However the non-uniqueness of such measure prevents to fix uniquely the price of European derivatives. Some practitioners have a positive view on this property of the trinomial model, since the freedom in choosing the parameter ω can be used to better calibrate the model. Moreover, regardless of which martingale measure one chooses, it is generally not possible to hedge European derivatives self-financially, that is to say, the trinomial model is incomplete (see Remark 0.8).

Task 1.3. *Consider a one-period trinomial model with $u = -d \neq 0$ and a derivative with pay off $Y = g(S(1))$. Show that a (constant) portfolio (h_S, h_B) hedging the derivative exists if and only if*

$$g(S_0 e^{-u}) - g(S_0)e^{-u} - g(S_0) + g(S_0 e^u)e^{-u} = 0.$$

Deduce from here that, for instance, the call option with strike $K = S_0$ cannot be hedged.

Incomplete models, of which the trinomial model is just an example, are investigated extensively by scholars and the community is divided among those who believe that incomplete models should be rejected and others who instead believe that real markets are incomplete and therefore incomplete models are more realistic.

1.2 Pricing options in incomplete markets

In the trinomial model there exist infinitely many martingale probabilities (q_u, q_m, q_d). Each martingale measure gives rise to a different price for the European derivative with pay-off Y at maturity $T = N$; denoting by $\mathbb{E}_\omega$ the expectation in the probability measure (1.6) and by $\Pi_Y(t, \omega)$ the price of the derivative derived from this measure, we have

$$\Pi_Y(t, \omega) = e^{-r(N-t)}\mathbb{E}_\omega[Y|S(1), \dots, S(t)].$$

Task 1.4. *Prove the recurrence formula* $\Pi_Y(N, \omega) = Y$,

$$\Pi_Y(t, \omega) = e^{-r}[q_u\Pi_Y^u(t+1, \omega) + q_m\Pi_Y^m(t+1, \omega) + q_d\Pi_Y^d(t+1, \omega)], \quad t = 0, \dots, N-1. \tag{1.7}$$

In Task 1.5 below it is asked to compute $\Pi_Y(0, \omega)$ with Matlab using the recurrence formula (1.7). To simplify the analysis we assume that the parameters of the trinomial model are

$$u = -d, \quad 0 < r < u, \quad p_u = p_d = p \in (0, 1/2). \tag{1.8}$$

Thus (1.3) becomes

$$S(t) = S(t-1)e^{uX_t}, \quad X_t = \begin{cases} -1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-2p \\ 1 & \text{with prob. } p \end{cases}. \tag{1.9}$$

Moreover, according to Task 1.2, for each value

$$0 < \omega < \frac{e^u - e^r}{e^u - 1} := \omega_{\max}(r, u),$$

we have the martingale probability defined by

$$q_u = \frac{e^r - e^{-u}}{e^u - e^{-u}} - \omega\frac{e^{-u/2}}{e^{u/2} + e^{-u/2}}, \quad q_m = \omega, \quad q_d = \frac{e^u - e^r}{e^u - e^{-u}} - \omega\frac{e^{u/2}}{e^{u/2} + e^{-u/2}}. \tag{1.10}$$

Now let $0 = t_0 < t_1 < \dots < t_N = T$ be a uniform partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$. Define $S(0) = S_0$ and

$$S(t_i) = S(t_{i-1})e^{uX_i}, \quad i = 1, \dots, N, \tag{1.11}$$

where the random variables $X_1, \dots, X_N$ are given by (1.9). The instantaneous variance of the stock is defined, as for the binomial model, by

$$\sigma^2 = \frac{1}{h}\mathrm{Var}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{2}{h}pu^2. \tag{1.12}$$

Having chosen $u = -d$, the instantaneous mean of log return is zero. The interest rate on each period becomes rh and, according to (1.12), $u = \sqrt{\frac{h}{2p}}\sigma$. It is easy to see that

$$\omega_{\max}\Big(rh, \sqrt{\frac{h}{2p}}\sigma\Big) \to 1, \text{ as } h \to 0^+.$$

Hence provided h is sufficiently small we can assume that $0 \leq \omega \lesssim 1$. Moreover the recurrence formula (1.7) becomes $\Pi_Y(t_N, \omega) = Y$, and

$$\Pi_Y(t_i, \omega) = e^{-rh}[q_u \Pi^u_Y(t_{i+1}, \omega) + q_m \Pi^m_Y(t_{i+1}, \omega) + q_d \Pi^d_Y(t_{i+1}, \omega)], \quad i = 0, \ldots, N-1. \tag{1.13}$$

Task 1.5 (Matlab)**. Part I.** *Write a Matlab function that computes the trinomial price at time $t = 0$ of the European call option with strike K and maturity T when $\omega \in (0, 1)$ is fixed. Show numerically that the result depends on the physical probability p. Plot the curves $\omega \to \Pi_Y(0, \omega)$ for different values of p and show numerically that the binomial and the trinomial price converge to the same value as $N \to \infty$ (namely, the Black-Scholes price) only for a specific value $\omega = \omega(p)$. For this value of ω, study the speed of convergence to the Black-Scholes price as $N \to \infty$ for different values of $p \in (0, 1/2)$ and show that the trinomial model converges to the Black-Scholes price faster than the binomial model.*

Part II. *Verify whether it is possible to use the parameter ω to replicate the market value of call options on S&P 500 with fixed maturity and different strikes, without changing the volatility parameter σ. How would you interpret this result? TIPS: use the 20days historical volatility of S&P 500 (easy to find on the Internet) for the value of σ and set $r = 0$. The option chain of S&P 500 can be found e.g. by googling "S&P 500 option chain Yahoo finance". Choose only options nearly at the money, say the first 10 options out of the money and the first 10 options in the money.*

Chapter 2

A project on forward and futures contracts

The purpose of this project is to model the evolution of the forward and future price of an asset. We begin by discussing in some details the properties of forward and futures contracts.

2.1 Forward and Futures

Forward contracts

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a European type financial derivative stipulated by two parties in which one promises to the other to sell (and possibly deliver) the asset $\mathcal{U}$ at time T in exchange for the cash K. As opposed to option contracts, both parties in a forward contract are *obliged* to fulfill their part of the agreement. In particular, as they both have the same right/obligation, neither of the two parties has to pay a premium to the other when the contract is stipulated, that is to say, *forward contracts are free*; in fact, the terminology used for forward contracts is "to enter a forward contract" and not "to buy/sell a forward contract". The party who must sell the asset at maturity is said to hold the short position, while the party who must buy the asset is said to hold the long position on the forward contract, although this terminology refers more precisely to the position on the underlying asset. Hence the pay-off for a long position in a forward contract on the asset $\mathcal{U}$ is

$$Y_{\text{long}} = (\Pi^{\mathcal{U}}(T) - K),$$

while for the holder of the short position the pay-off is

$$Y_{\text{short}} = (K - \Pi^{\mathcal{U}}(T)).$$

Forward contracts are traded OTC and most commonly on commodities or market indexes, such as currency exchange rates, interest rates and volatilities. In the case that the underlying asset is an index, forward contracts are also called **swaps** (e.g., currency swaps, interest rate swaps, volatility swaps, etc.).

One purpose of forward contracts is to share risks. Irrespective of the movement of the underlying asset in the market, its price at time T for the holders of the forward contract will be K. The delivery price agreed by the two parties in a forward contract is also called the **forward price** of the asset. More precisely, the T-forward price $\mathrm{For}_{\mathcal{U}}(t,T)$ of an asset $\mathcal{U}$ at time $t<T$ is the strike price of a forward contract on $\mathcal{U}$ stipulated at time t and with maturity T, while the current, actual price $\Pi^{\mathcal{U}}(t)$ of the asset is called the **spot price**. Note that the forward price $\mathrm{For}_{\mathcal{U}}(t,T)$ is unlikely to be a good estimation for the price of the asset at time T, since the consensus on this value is limited to the participants of the forward contract and different parties may agree to different delivery prices. The delivery price of futures contracts on the asset, which we define in the next section, gives a better and more commonly accepted estimation for the future value of an asset.

Theorem 2.1. *Suppose that the market $\{\Pi^{\mathcal{U}}(t), B(t)\}_{t=0,\dots,N}$ is complete. Show that the arbitrage-free forward price of the asset $\mathcal{U}$ for delivery at time T is given by*

$$\mathrm{For}_{\mathcal{U}}(t,T)=\frac{\Pi^{\mathcal{U}}(t)}{B(t,T)}, \tag{2.1}$$

where $B(t,T)$ is the risk-neutral price at time t of the ZCB with face value 1 and expiring at time T.

Task 2.1. *Prove the theorem. HINT: You need the put-call parity.*

Task 2.2. *Compute the forward price of the asset at all times $t=0,1,2,3$ in the example of market at the end of Section 0.4.*

Futures

Futures are standardized forward contracts, i.e., rather than being OTC, they are negotiated in regularized markets. Specifically, a **futures market** is a market in which the object of trading are futures contracts. Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation, and so in particular all contracts on the same asset with the same time of maturity T have the same delivery price. The T-**future price** $\mathrm{Fut}_{\mathcal{U}}(t,T)$ of the asset $\mathcal{U}$ at time $t\leq T$ is defined as the delivery price at time $t\leq T$ in the futures contract on the asset $\mathcal{U}$ with maturity T. Futures markets have been existing for more than 300 years and nowadays the most important ones are the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the International Exchange Group (ICE).

In a futures market, anyone (after a proper authorization) can stipulate a futures contract. More precisely, holding a position in a futures contract in the futures market consists in the

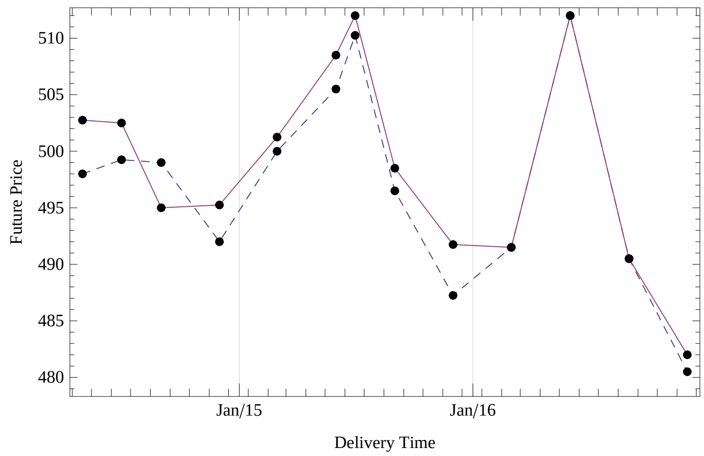


Figure 2.1: Futures price of corn on May 12, 2014 (dashed line) and on May 13, 2014 (continuous line) for different delivery times

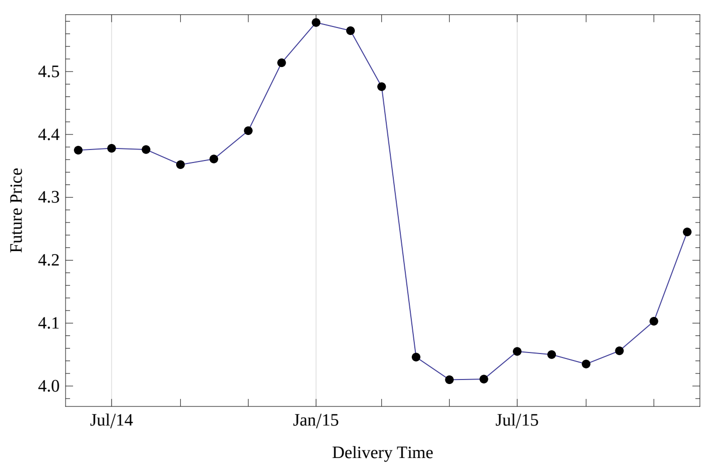


Figure 2.2: Futures price of natural gas on May 13, 2014 for different delivery times

agreement to receive as a cash flow the change in the future price of the underlying asset during the time in which the position is held. The cash flow may be positive or negative. In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down. Moreover, in order to alleviate the risk of insolvency, the cash flow is distributed in time through the mechanism of the **margin account**. For example, assume that at $t = 0$ we open a long position in a futures contract expiring at time T. At the same time, we need to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened). At $t = 1$ day, the amount $\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)$ will be added to the account, if it positive, or withdrawn, if it is negative. The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$(\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(t-1,T)) + (\mathrm{Fut}_{\mathcal{U}}(t-1,T) - \mathrm{Fut}_{\mathcal{U}}(t-2,T)) + \\ \cdots + (\mathrm{Fut}_{\mathcal{U}}(1,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)) = (\mathrm{Fut}_{\mathcal{U}}(t,T) - \mathrm{Fut}_{\mathcal{U}}(0,T)).$$

If a long position is held up to the time of maturity, then the holder of the long position should buy the underlying asset. However futures contracts are often **cash settled** and not **physically settled**, which means that the delivery of the underlying asset does not occur, and the equivalent value in cash is paid instead.

Our next purpose is to introduce the definition of arbitrage-free future price of an asset. Our strategy is to show is that any reasonable definition should satisfy 3 standard conditions and then show that these conditions define uniquely the future price as a stochastic process.

For simplicity we argue under the assumption that the underlying asset $\mathcal{U}$ and the money market make up a complete market of the form (42) with $S(t) = \Pi^{\mathcal{U}}(t)$ and $T = N$. As the generalized random walk $\{M_t\}_{t=0,\dots,N}$ contains all the information about the state of the market, we are naturally led to impose the following first condition on the future price.

Assumption 1. The future price process $\{\mathrm{Fut}_{\mathcal{U}}(t,T)\}_{t=0,\dots,T=N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$.

For the next assumption we need to derive a recurrence formula for the value of portfolio processes invested in a futures contract and in the money market (similar to the formula (31) for the value of self-financing portfolios invested in the stock and the risk-free asset). Consider a portfolio process that, at time $t < T$, consists of $h(t)$ shares of the futures contract expiring at time T and $h_{t+1}(t)$ shares of the ZCB with pay-off 1 maturing at time $t+1$. As the ZCB has very short time left to maturity, then $h_t(t+1)$ is our position on the money market (recall that the money market consists of short term loan assets). We assume that the portfolio process is predictable from $\{M_t\}_{t=0,\dots,N}$. As futures contracts have zero value, the value of the portfolio at time t is simply the money market account:

$$V(t) = h_{t+1}(t)B(t,t+1) = \frac{h_{t+1}(t)}{1+R(t)}.$$

At time $t+1$ the owner of the portfolio will receive the pay-off of the ZCB and the change in value of the future price (multiplied by the number of shares). Hence at time $t+1$ the portfolio generates the cash flow

$$\begin{aligned} C(t+1) &= h_{t+1}(t) + h(t)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)) \\ &= V(t)(1+R(t)) + h(t)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)). \end{aligned}$$

In a portfolio invested in futures and ZCB's this cash should be immediately re-invested in the money market (this is the only possibility, as changing position on the futures contract costs nothing). Hence $C(t+1) = h_{t+2}(t+1)B(t+1,t+2) = V(t+1)$. It follows that

$$\begin{aligned} h(t)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)) &= V(t+1) - (1+R(t))V(t) = V(t+1) - \frac{D(t)}{D(t+1)}V(t) \\ &= D(t+1)^{-1}[D(t+1)V(t+1) - D(t)V(t)]. \end{aligned}$$

Thus the value of any portfolio process invested in the futures contract and the money market satisfies

$$h(t)D(t+1)(\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)) = V^*(t+1) - V^*(t), \tag{2.2}$$

where $V^*(t) = D(t)V(t)$ is the discounted (at time $t = 0$) portfolio value. Now, by the arbitrage-free principle, this portfolio should not be an arbitrage. We have seen that this condition can be achieved by imposing that the discounted value of the portfolio processes is a martingale (see the proof of Theorem (0.6)). This holds in particular if

$$\widetilde{\mathbb{E}}[V^*(t+1)|M_0,\dots,M_t] = V^*(t),$$

for all $t = 0,\dots,T-1$. Hence, taking the conditional expectation of both sides of (2.2) with respect to $M_0,\dots,M_t$, we obtain

$$h(t)D(t+1)\widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[V^*(t+1) - V^*(t)|M_0,\dots,M_t] = 0,$$

where we used that $h(t)$ and $D(t+1)$ are measurable with respect to $M_0,\dots,M_t$ and thus can be taken out from the conditional expectation. By Assumption 1 we have

$$\widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T) - \mathrm{Fut}_{\mathcal{U}}(t,T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T)|M_0,\dots,M_t] - \mathrm{Fut}_{\mathcal{U}}(t,T).$$

Hence the market is free of arbitrages if we assume the following.

Assumption 2. The future price satisfies

$$\widetilde{\mathbb{E}}[\mathrm{Fut}_{\mathcal{U}}(t+1,T)|M_0,\dots,M_t] = \mathrm{Fut}_{\mathcal{U}}(t,T), \quad t = 0,\dots,T-1.$$

The last natural assumption is that the future price at maturity $t = T$ should coincide with the spot price $\Pi(T)$ of the asset, i.e.,

Assumption 3. $\mathrm{Fut}_{\mathcal{U}}(T,T) = \Pi(T)$.

Theorem 2.2. *There is only one stochastic process $\{\mathrm{Fut}_{\mathcal{U}}(t,T)\}_{t=0,\dots,T}$ that satisfies Assumptions 1-3, namely*

$$\mathrm{Fut}_{\mathcal{U}}(t,T) = \widetilde{\mathbb{E}}[\Pi(T)|M_0,\dots,M_t]. \tag{2.3}$$

Task 2.3. *Prove the theorem.*

Task 2.4. *Compute the futures price of the asset at all times $t = 0,1,2,3$ in the example of market given at the end of Section 0.4.*

2.2 Computation of the futures price with Matlab

The purpose of this section is to compute numerically the future price at time $t = 0$ of an asset in the market (42) with the risk-free rate given by the Ho-Lee model (45). To this purpose we first have to make a choice for the physical probability. We assume that *in the real world the transition probabilities* (41) *are constant and given by* $p_t \equiv 1/2$. This implies in particular that $\{M_t\}_{t=0,\dots,N}$ is a standard symmetric random walk, and thus that the asset price is binomially distributed, in the physical probability[1]. Now, let $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a uniform partition of the interval $[0, T]$ with size $h = t_i - t_{i-1}$ and assume $u = -d$ for simplicity. The mean of log-return of the underlying asset is therefore zero and

$$u = \sqrt{h}\,\sigma, \tag{2.4}$$

where σ is the volatility of the underlying asset, see (39) (with $\alpha = 0$ and $p = 1/2$). The risk-free rate in the time interval $[t_{i-1}, t_i]$ is now given by $hR(i)$, where $R(i)$ follows the Ho-Lee model $R(i) = a(i) + b(i)M_i$, $i = 0, \dots, N-1$, see (45). The risk-neutral transition probability in the interval $[t_{i-1}, t_i]$ now takes the form

$$q_i(k) = \frac{1 + h(a(i-1) + b(i-1)k) - e^{-u}}{e^u - e^{-u}}, \quad k = -i+1, -i+3, \dots i-1,$$

see (48) (with the substitution $(a, b, d) \to (ha, hb, -u)$). For simplicity we work with the functions $a(k)$, $b(k)$ given by (50), hence

$$q_1(0) = \frac{1 + a_0 h - e^{-u}}{e^u - e^{-u}}, \quad q_i(k) = \frac{1 + h(a_0 + \frac{b_0 k}{i-1}) - e^{-u}}{e^u - e^{-u}},$$

see (52). This choice is admissible provided a_0, b_0, u satisfy

$$a_0 h > b_0 h - 1, \quad e^{-u} < 1 + h(a_0 - b_0), \quad e^u > 1 + h(a_0 + b_0),$$

see (51). It is straightforward to verify that these inequalities hold when h is sufficiently small. By Theorem 2.2 the future price at time $t = 0$ of the asset is $\mathrm{Fut}_{\mathcal{U}}(0, T) = \widetilde{\mathbb{E}}[\Pi(T)]$. Hence, using Theorem 0.3, we have

$$\mathrm{Fut}_{\mathcal{U}}(0, T) = \widetilde{\mathbb{E}}[\Pi^{\mathcal{U}}(T)] = \sum_{x \in \{-1,1\}^N} \widetilde{\mathbb{P}}(x) \Pi^{\mathcal{U}}(T, x) \tag{2.5}$$

where

$$\Pi^{\mathcal{U}}(T, x) = \Pi^{\mathcal{U}}(0) \exp\left((x_1 + \cdots + x_N)u\right)$$

is the asset price at time T along the path $x \in \{-1, 1\}^N$ and $\widetilde{\mathbb{P}}(x)$ is the risk-neutral probability of realization of the path x, which is computed according to (23) (with $p_t \equiv q_t$).

[1] In the applications of the model, this assumption affects the calibration of the parameters. The assumed distribution of the asset price in the physical probability must be properly tested using historical data.

Now, while it is not hard to implement the formula (2.5) with Matlab, it is clear that this is possible only for a relatively small number of steps (up to, say, $N \approx 20$), for one needs to create all 2^N elements of the set $\{-1,1\}^N$ in order to compute the sum in the right hand side of (2.5). Dealing with a large number of steps is possible by using the discrete version of the crude Monte Carlo method presented in Section (0.7). More precisely, letting $\mathcal{O}$ be a set of M *randomly chosen* paths of $\{M_t\}_{t=0,\dots,N}$, we approximate the value of $\mathrm{Fut}_{\mathcal{U}}(0,T)$ by

$$\mathrm{Fut}_{\mathcal{U}}(0,T) \approx F = \frac{2^N}{M} \sum_{x \in \mathcal{O}} \widetilde{\mathbb{P}}(x) \Pi^{\mathcal{U}}(T,x), \tag{2.6}$$

that is to say, we restrict the sum in (2.5) to the paths in the set $\mathcal{O}$ and multiply further by the factor $2^N/M$, which is the total number of paths divided by the number of sample paths. In order to measure the accuracy of this approximation, we begin by repeating the above calculation n times (using every time a different set of M sample paths) to produce the approximations $F(1), \dots, F(n)$ for the future price of the asset and pick, as our best estimate of its exact value, the sample average

$$\overline{F} = \frac{1}{n} \sum_{i=1}^{n} F(i).$$

To measure how reliable is the approximation $\overline{F}$, we compute the so called **standard error of the mean**

$$\mathrm{Err} = \frac{s}{\sqrt{n}}, \quad \text{where} \quad s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (F(i) - \overline{F})^2}$$

is the standard deviation of the sample $F(1), \dots, F(n)$.

Task 2.5 (Matlab)**. Part I.** *Write a Matlab code that computes the crude Monte Carlo approximation F of the future price as explained above. Show in two pictures how $\overline{F}$ and* Err *depend on the number of paths using, say, $n = 50$ trials.*

Part II. *Plot the Future price curve $T \to \mathrm{Fut}_{\mathcal{U}}(0,T)$. Show how this curve depends on the volatility parameters b_0 and σ and explain your findings.*

Remark 2.1. The implementation of the algorithm in the previous exercise is quite demanding from a computer power point of view. Start with a relatively small number of steps, say 1000, and see how far you can push this number without making the calculation too long (1000000 should be within reach of last generation computers). Moreover the results are more precise when low volatilities b_0, σ are used (why?).

Chapter 3

A project on the Asian option

The risk-neutral pricing formula for European call and put options, and for other simple standard European derivatives, reduces to a simple expression involving the standard normal distribution, see (78) for the case of European calls. For Asian options, and other path-dependent options, this reduction is not possible, and the application of numerical methods to valuate these derivatives becomes essential. The Monte Carlo numerical method is the most popular among practitioners. This project deals with applications of the Monte Carlo method to compute the risk-neutral value of Asian options.

The Asian option

The Asian call/put option in the time-continuum case is defined as the non-standard European derivative with pay-off

$$Y^{\text{call}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+, \quad Y^{\text{put}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+,$$

where $K > 0$ is the strike price of the option. The Black-Scholes price at time $t = 0$ of these options is given by

$$\Pi_{\text{AC}}(0) = e^{-rT}\mathbb{E}_q[Y^{\text{call}}], \quad \Pi_{\text{AP}}(0) = e^{-rT}\mathbb{E}_q[Y^{\text{put}}].$$

Task 3.1. *Derive the following put-call parity identity:*

$$\Pi_{\text{AC}}(0) - \Pi_{\text{AP}}(0) = e^{-rT}\left(\frac{e^{rT}-1}{rT}S_0 - K\right). \tag{3.1}$$

Task 3.2. *Let C_0 denote the Black-Scholes price at time $t = 0$ of the standard European call option with strike K and maturity T. Prove the inequality*

$$\Pi_{\text{AC}}(0) \le \frac{1-e^{-rT}}{rT}C_0, \tag{3.2}$$

for the Asian option on the same stock and with the same strike and maturity of the European one. HINT: You need the so called Jensen's inequality for integrals:

$$\phi\Big(\frac{1}{b-a}\int_a^b f(x)\,dx\Big) \leq \frac{1}{b-a}\int_a^b \phi(f(x))\,dx,$$

for all real-valued convex functions ϕ. You also need to remember that the value of the European call increases with maturity.

Remark 3.1. Since the function $x \to (1-e^{-x})/x$ is bounded by one for $x \geq 0$, the inequality (3.2) implies that for $r \geq 0$ the Asian call is cheaper than the European one.

Task 3.3. *The Asian call with geometric average is the non-standard European derivative with pay-off*

$$Q = \left(e^{\frac{1}{T}\int_0^T \log S(t)\,dt} - K\right)_+. \tag{3.3}$$

Show that the Black-Scholes price at time $t=0$ of this derivative is given by

$$\Pi_{\mathrm{AC}}^{(\mathrm{G})}(0) = e^{-rT}(e^{qT}S_0\Phi(d_1) - K\Phi(d_2)) \tag{3.4a}$$

where

$$q = \frac{1}{2}(r - \frac{\sigma^2}{6}), \quad d_2 = d_1 - \sigma\sqrt{\frac{T}{3}}, \quad d_1 = \frac{\log\frac{S_0}{K} + \frac{1}{2}(r+\frac{\sigma^2}{6})T}{\sigma\sqrt{T/3}}.$$

Derive also the anlogous formula for the Asin put with geometric average and the corresponding put-call parity. HINT: You need to apply Theorem 0.11.

Monte Carlo valuation of the Asian option

Letting $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0,T]$ with size $t_i - t_{i-1} = h$. We approximate the pay-off of the Asian option on the given partition as

$$Y = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+ \approx \left(\frac{1}{N}\sum_{i=1}^N S(t_i) - K\right)_+.$$

Task 3.4 (Matlab)**. Part I.** *Write a Matlab code that computes the Black-Scholes price at time $t=0$ and the confidence interval of the Asian option (call and put) using the crude Monte Carlo method. Write also a code which applies the control variate Monte Carlo method using the pay-off of the Asian option with geometric mean as control variate. Compare the new method with the crude Monte Carlo method and show that the control variate technique improves the performance of the computation. Use the put-call parity to verify the accuracy of your numerical findings.*

Part II. *Use the control variate Monte Carlo method to study numerically how the price of the Asian call and put depend on the parameters of the option. Plot the curves $\lambda \to \Pi_{\mathrm{AC}}(0)$, where λ is any of the parameters σ, r, K, T, S_0. Show in each plot the analogous curve for the standard European call and use your results to discuss the main differences between the European call and the Asian call options.*

Chapter 4

A project on coupon bonds

Coupon bonds are debt instruments issued by national governments as a way to borrow money and fund their activities. Given the long maturity of coupon bonds (which can reach up to 30 or more years), the valuation of these contracts must take into account the time fluctuations of the risk-free rate. Once a stochastic model for the risk-free rate is prescribed, the valuation of coupon bonds can be carried out using the so called "classical approach", which is based on the risk-neutral pricing formula. The main purpose of this project is to numerically compute the yield curve of coupon bonds implied by a particular example of stochastic risk-free rate model.

4.1 Zero-coupon and coupon bonds

A **zero-coupon bond** (ZCB) with **face** (or **nominal**) value K and maturity $T > 0$ is a contract that promises to pay to its owner the amount K at time T in the future. Zero-coupon bonds, and the related coupon bonds described below, are issued by national governments and private companies as a way to borrow money and fund their activities. Without loss of generality we assume from now on that $K = 1$, as owning a ZCB with face value K is clearly equivalent to own K shares of a ZCB with face value 1. Moreover in the following we assume that all ZCB's are issued by one given institution, so that all bonds differ merely by their maturities.

After being originally issued in the so-called **primary market**, the ZCB becomes a tradable asset in the **secondary** bond market. It is therefore natural to model the value at time t of the ZCB maturing at time $T > t$ as a random variable, which we denote by $B(t,T)$. Hence $\{B(t,T)\}_{t\in[0,T]}$ is a stochastic process. We assume throughout the discussion that the institution issuing the bond bears no risk of default, i.e., $B(t,T) > 0$, for all $t \in [0,T]$. Clearly $B(T,T) = 1$ and, under normal market conditions, $B(t,T) < 1$, for $t < T$, i.e., ZCB's are risk-free assets ensuring a positive return. However exceptions are possible; for instance national bonds in Sweden with maturity shorter than 5 years yield currently (2017) a negative return. A **ZCB market** is a market that consists of ZCB's with different maturities. Our main goal is to introduce models for the price of ZCB's observed in the market. For modeling purposes we assume that there is a ZCB in the market maturing at each time $T \in [0, S]$, where S is the maturity of the latest expiring ZCB in the market (e.g., $S \approx 30$ years). Note that this assumption is quite far from reality, one reason being that bonds with maturity larger than, say, 2 years will most likely pay coupons.

Interest rates and yield of ZCB's

The difference in value of ZCB's with different maturities is expressed through the implied forward rate of the bond. To define this concept, suppose that at the present time t we open a portfolio that consists of -1 share of a ZCB with maturity $t < T$ and $B(t,T)/B(t,T+\delta)$ shares of a ZCB expiring at time $T+\delta$. This investment has zero value and entails that we pay 1 at time T and receive $B(t,T)/B(t,T+\delta)$ at time $T+\delta$. Hence our investment at the present time t is equivalent to an investment in the future time interval $[T, T+\delta]$ with (annualized) return given by

$$F_\delta(t,T) = \frac{1}{\delta}\left(B(t,T)/B(t,T+\delta) - 1\right) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)}. \tag{4.1}$$

The quantity $F_\delta(t,T)$ is also called **discretely compounded forward rate** in the interval $[T, T+\delta]$ **locked at time** t (or forward LIBOR, as it is commonly applied to LIBOR interest rate contracts). The name is intended to emphasize that the investment return in the future interval $[T, T+\delta]$ is locked at the *present* time $t \leq T$, that is to say, we know today which interest rate has to be charged to borrow in the future time interval $[T, T+\delta]$ (if a different rate were locked today, then an arbitrage opportunity would arise). When $\delta \to 0^+$ we obtain the **continuously compounded T-forward rate**

$$f(t,T) = \lim_{\delta \to 0^+} \frac{1}{\delta} \frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)} = -\partial_T \log B(t,T), \tag{4.2}$$

which is the rate locked at time t to borrow at time T for an "infinitesimal" period of time. In the following we shall consider only continuously compounded rates.

The curve $T \to f(t,T)$ is called **forward rate curve** of the ZCB market. The knowledge of the forward rate curve determines the price $B(t,T)$ of all ZCB's in the market through the formula

$$B(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right), \quad 0 \leq t \leq T \leq S, \tag{4.3}$$

which follows easily by integrating (4.2).

The quantity

$$r(t) = f(t,t), \quad t \in [0, S] \tag{4.4}$$

is called the **(continuously compounded) spot rate** of the ZCB market at time t and represents the interest rate locked at time t to borrow instantaneously at time t (i.e., on the spot).

The spot rate can be used to define the **discount process**:

$$d(t) = \exp\left(-\int_0^t r(s)\,ds\right) \quad \text{(continuously compounded).} \tag{4.5}$$

If t is the present time and $X(\tau)$ is the value of an asset at some given future time $\tau > t$, then the quantity

$$\frac{d(\tau)}{d(t)} X(\tau) = \exp\left(-\int_t^\tau r(s)\,ds\right) X(\tau)$$

is called the present (at time t) **discounted value** of the asset and represents the future (at time τ) value of the asset relative to the purchasing value of money at that time.

The (continuously compounded) **yield (to maturity)** $y(t,T)$ at time t of the ZCB with maturity T is the *constant* forward rate which entails the value $B(t,T)$ of the ZCB. Hence the yield $y(t,T)$ of a ZCB is obtained by replacing $f(t,v) = y(t,T)$ in (4.3), i.e.,

$$B(t,T) = e^{-y(t,T)(T-t)}, \quad \text{that is,} \quad y(t,T) = -\frac{\log B(t,T)}{T-t}. \tag{4.6}$$

To put it in other words: Selling a ZCB for the price $B(t,T)$ at time t (i.e., borrowing $B(t,T)$ at time t) is equivalent to lock the constant forward rate $y(t,T)$ until maturity. Note also that the first equation in (4.6) expresses $B(t,T)$ as the discounted value at time t of the future payment $= 1$ at maturity assuming that the spot rate is constant and equal to $y(t,T)$ in the interval $[t,T]$.

Coupon bonds

Let $0 < t_1 < t_2 < \cdots < t_M = T$ be a partition of the interval $[0,T]$. A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \ldots, c_M \in (0,1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1+c_M$ at maturity $T = t_M$. We set $c = (c_1, \ldots, c_M)$ and denote by $B_c(t,T)$ the value at time t of the bond paying the coupons $c_1, \ldots, c_M$ and maturing at time T. Now, let $t \in [0,T]$ and $k(t) \in \{1, \ldots, M\}$ be the smallest index such that $t_{k(t)} > t$, that is to say, $t_{k(t)}$ is the first time after t at which a coupon is paid. Holding the coupon bond at time t is clearly equivalent to holding a portfolio containing $c_{k(t)}$ shares of the ZCB expiring at time $t_{k(t)}$, $c_{k(t)+1}$ shares of the ZCB expiring at time $t_{k(t)+1}$, and so on, hence

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j B(t,t_j) + (1 + c_M) B(t,T), \tag{4.7}$$

the sum being zero when $k(t) = M$.

The yield of a coupon bond is defined implicitly by the equation

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j e^{-y_c(t,T)(t_j-t)} + (1+c_M)e^{-y_c(t,T)(T-t)} \tag{4.8}$$

and so *the yield of the coupon bond is the constant spot rate used to discount the total future payments of the coupon bond.*

Remark 4.1. Most commonly the coupons are equal. Letting $c_j = c$, for all $j = 1,\dots,M$, the formula (4.8) simplifies to

$$B_c(t,T) = c\sum_{j=k(t)}^{M-1} e^{-y_c(t,T)(t_j-t)} + (1+c)e^{-y_c(t,T)(T-t)}. \tag{4.9}$$

To compute the yield of a coupon bond with value $B_c(t,T)$, one has to invert (4.9). For instance, assume that $T = M$ years and that the coupons are paid annually, that is $t_1 = 1$, $t_2 = 2,\dots,t_M = M$. Then $x = e^{-y_c(0,T)}$ solves $p(x) = 0$, where p is the M-order polynomial given by

$$p(x) = c_1x + c_2x^2 + \cdots + (1+c_M)x^M - B_c(0,T). \tag{4.10}$$

The roots of this polynomial can easily be computed numerically, e.g., with the command `roots[p]` in matlab, see Task 4.4 below.

Yield curve

(Zero) coupon bonds are listed in the market in terms of their yield rather than in terms of their price. The curve $T \to y_c(t,T)$ is called the **yield curve** of the ZCB market. Figure 4.1 shows an example of yield curve for governmental Swedish bonds.

Task 4.1. *Yield curves observed in the market are classified based on their shape (e.g., steep, flat, inverted, etc.). Find out on the Internet what the different shapes mean from an economical point of view and write a short text about this.*

4.2 The classical approach to ZCB's pricing

In this section we describe the so-called **classical approach** to ZCB's pricing. This approach is based on the risk-neutral pricing formula.

Definition 4.1. *Let $\{r(t)\}_{t\geq 0}$ be a stochastic process modeling the spot interest rate of the ZCB market, where we assume that $r(0) = r_0$ is a deterministic constant. Then*

$$B(0,T) = \mathbb{E}[d(T)] = \mathbb{E}[e^{-\int_0^T r(s)\,ds}] \tag{4.11}$$

is called the risk-neutral price of the ZCB at time $t = 0$.

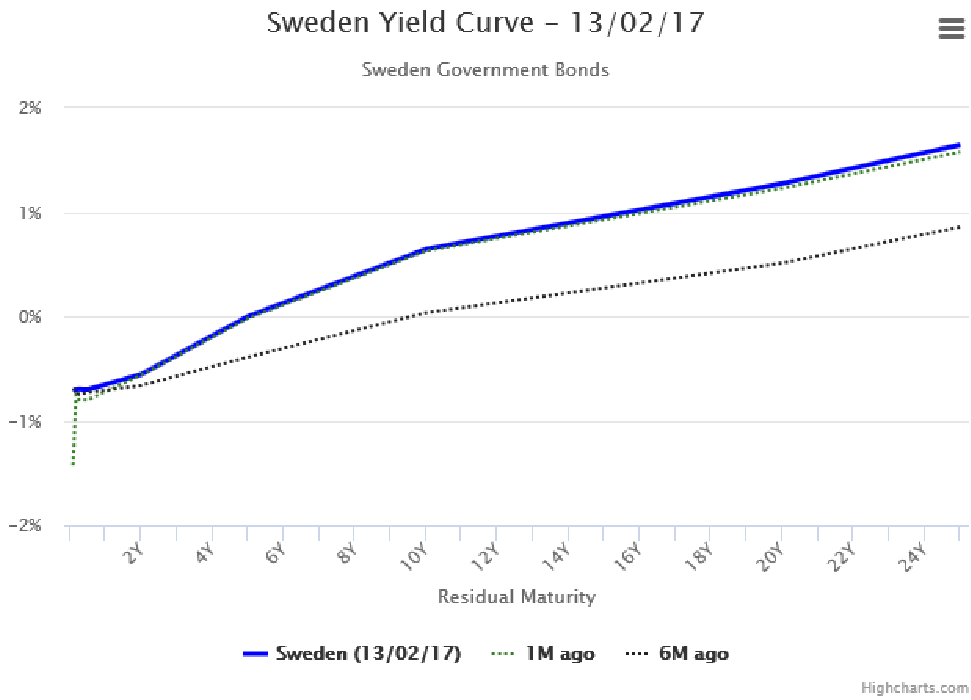


Figure 4.1: Yield curve for Swedish bonds. Note that the yield is negative for maturities shorter than 5 years. Bonds with maturity larger than 2 years have coupon and thus their yield is computed using (4.8) (instead of (4.6)).

Hence the value at time $t = 0$ of the ZCB is the expected value of the discounted future payment $= 1$ of the ZCB. Note that in a purely ZCB market, one cannot define a martingale, or risk-neutral, probability, hence the expectation in (4.11) is taken in the physical probability. Equivalently, in the classical approach to ZCB's pricing, the physical probability is assumed to be risk-neutral.

In the following two tasks it is asked to compute the exact value of $B(0,T)$ when the spot rate is given by two simple models, namely the Ho-Lee model and the Vasicek model.

Task 4.2. *In the* **Ho-Lee model**, *the risk-free rate is assumed to satisfy*

$$r(t) = r(0) + \theta(t) + \sigma W(t), \quad \theta(0) = 0, \tag{4.12}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion, $\sigma > 0$ is constant and $\theta(t)$ is a deterministic function of time. Derive the initial price $B(0,T)$ of the ZCB with face value 1 and maturity $T > 0$ and the forward rate $f(0,T)$ implied by the Ho-Lee model. Draw (e.g., with Maltab) the graph $T \to f(0,T)$ of the forward curve at time $t = 0$ (experiment for different functions θ). HINT: You need Theorem 0.11.

Task 4.3. *In the* **Vasicek model***, the risk-free rate is assumed to satisfy*

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma W(t) - a\sigma \int_0^t e^{a(s-t)} W(s)\, ds, \tag{4.13}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion and $a > 0, b \in \mathbb{R}, \sigma > 0$ are constants. Note that, by Theorem 0.11, $r(t)$ is normally distributed. Show that the initial price of the ZCB with face value 1 and maturity $T > 0$ is given by

$$B(0,T) = e^{-r(0)A(T)+C(T)}, \tag{4.14a}$$

where

$$A(T) = \frac{1}{a}(1 - e^{-aT}), \tag{4.14b}$$

$$C(T) = \left(b - \frac{\sigma^2}{2a^2}\right)(A(T) - T) - \frac{\sigma^2}{4a} A(T)^2. \tag{4.14c}$$

HINT: You need Theorem 0.11.

Remark 4.2. By using the notation in Remark 0.17, we can write the definition of $r(t)$ in the Vasicek model as

$$r(t) = r(0) + b(e^{at} - 1) + \sigma \int_0^t e^{as} dW(s),$$

which is the form of $r(t)$ most used in the literature.

Task 4.4 (Matlab)**. Part I.** *Write the code for a Matlab function*

```
yield(B, Coupon, CouponDates)
```

that computes the continuously compounded yield to maturity of a coupon bond. Here, `B` *is the current (i.e., at time $t = t_0 = 0$) price of the bond,* `Coupon` $\in (0,1)$ *is the (constant) coupon,* `CouponDates` *is the vector $(t_1, \dots, t_M = T)$ containing the dates at which the coupon is paid, the last component of which is the maturity of the bond. Remember that time in finance is measured in fraction of years and 1 year = 252 days (unless otherwise stated in the contract).*

Part II. *Let $\{r(t)\}_{t\geq 0}$ be given by the Vasicek model with parameters a, b, σ. Apply the code in Part I to perform a parameter sensitivity analysis of the yield curve. Can you reproduce all the typical shapes found in Exercise 4.1?*

Chapter 5

A project on multi-asset options

Multi-asset options are financial derivatives on several underlying assets. In this project we consider standard European derivatives on two stocks, i.e., European style derivatives with pay-off of the form $Y = g(S_1(T), S_2(T))$, where $S_1(t), S_2(t)$ are the prices of the underlying stocks at time $t \in [0, T]$ and $T > 0$ is the time of maturity of the derivative. After the Black-Scholes theory for single stock options is generalized to the two dimensional case, the Monte Carlo method is applied to compute the price of maximum call options on two stocks.

5.1 Examples of options on two stocks

Given $K_1, K_2 > 0$, a **two assets correlation** call option with maturity T is the European derivative with pay-off

$$Y = \begin{cases} (S_2(T) - K_2)_+ & \text{if } S_1(T) > K_1 \\ 0 & \text{otherwise} \end{cases} .$$

A **maximum** call option on two stocks with maturity T is the European style derivative with pay-off $Y = \max((S_1(T)-K_1)_+, (S_2(T)-K_2)_+)$, and similarly one defines the minimum call option on two stocks and the analogous put options.

The European derivative with maturity T and pay-off

$$Y = (S_1(T) - S_2(T))_+$$

is called a **spread** option (or **exchange asset** option). The European derivative with maturity T and pay-off

$$Y = \left(\frac{S_1(T)}{S_2(T)} - K \right)_+$$

is called a **relative outperformance** option.

A **quanto** option is a call or put option on a stock in which the pay-off is paid in a different currency than the one in which the stock is traded. Thus, letting $\Xi(t)$ be the exchange rate of the two currencies at time t, the pay-off of a quanto call option with maturity T is

$$Y = \Xi(T)(S(T) - K)_+.$$

Note that in this example the second asset is not at stock but a market index (the exchange rate $\Xi(t)$).

The list of examples could go on, but we stop here. New types of multi-asset options are created frequently. All multi-asset options are traded OTC.

5.2 Black-Scholes price of 2-assets standard European derivatives

In the Black-Scholes theory of two-dimensional markets it is assumed that the stocks prices are given by a 2-dimensional geometric Brownian motion, namely:

$$S_1(t) = S_1(0)e^{\alpha_1 t + \sigma_{11}W_1(t) + \sigma_{12}W_2(t)}, \tag{5.1a}$$
$$S_2(t) = S_2(0)e^{\alpha_2 t + \sigma_{21}W_1(t) + \sigma_{22}W_2(t)}, \tag{5.1b}$$

where $\{W_1(t)\}_{t\geq 0}, \{W_2(t)\}_{t\geq 0}$ are independent Brownian motions in the physical probability $\mathbb{P}$ and $\alpha_1, \alpha_2, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ are real constants. We assume that the **volatility matrix** $\sigma = (\sigma_{ij})$ is invertible. Letting

$$W(t) = (W_1(t), W_2(t)), \quad \sigma_1 = (\sigma_{11}, \sigma_{12}), \quad \sigma_2 = (\sigma_{21}, \sigma_{22}),$$

we can rewrite the 2-dimensional geometric Brownian motion in the more concise form

$$S_j(t) = S_j(0)e^{\alpha_j t + \sigma_j \cdot W(t)},$$

where $\cdot$ denotes the standard scalar product of vectors. We start by deriving the joint density of the stocks prices.

Theorem 5.1. *The random variables $\log S_1(t), \log S_2(t)$ are jointly normally distributed with mean $m = (\log S_1(0) + \alpha_1 t, \log S_2(0) + \alpha_2 t)$ and covariant matrix $C = t\sigma\sigma^T$. In particular, the random variables $S_1(t), S_2(t)$ have the joint density*

$$f_{S_1(t),S_2(t)}(x,y) = \frac{e^{-\frac{1}{2t}\left(\log\frac{x}{S(0)} - \alpha_1 t \quad \log\frac{y}{S(0)} - \alpha_2 t\right)(\sigma\sigma^T)^{-1}\begin{pmatrix}\log\frac{x}{S(0)} - \alpha_1 t \\ \log\frac{y}{S(0)} - \alpha_2 t\end{pmatrix}}}{txy\sqrt{(2\pi)^2\det(\sigma\sigma^T)}}. \tag{5.2}$$

Proof. We have

$$\log S_1(t) = \log S_1(0) + \alpha_1 t + \sigma_{11} W_1(t) + \sigma_{12} W_2(t),$$
$$\log S_2(t) = \log S_2(0) + \alpha_2 t + \sigma_{21} W_1(t) + \sigma_{22} W_2(t),$$

hence the first statement of the theorem follows by Theorem 0.10. The joint density of $S_1(t), S_2(t)$ is computed using that

$$\begin{aligned} F_{S_1(t),S_2(t)}(x,y) &= \mathbb{P}(S_1(t) \le x, S_2(t) \le y) \\ &= \mathbb{P}(\log S_1(t) \le \log x, \log S_2(t) \le \log y) = F_{\log S_1(t), \log S_2(t)}(\log x, \log y), \end{aligned}$$

hence

$$\begin{aligned} f_{S_1(t),S_2(t)}(x,y) &= \partial_x \partial_y F_{S_1(t),S_2(t)}(x,y) = \partial_x \partial_y [F_{\log S_1(t),\log S_2(t)}(\log x, \log y)] \\ &= (xy)^{-1} f_{\log S_1(t),\log S_2(t)}(\log x, \log y), \end{aligned}$$

which, using the joint normal density of $\log S_1(t)$ and $\log S_2(t)$, gives (5.2). □

The 2-dimensional geometric Brownian motion is often given in a different but equivalent (in distribution) form, as shown in the next Task.

Task 5.1. *Show that the process* (5.1) *is equivalent, in distribution, to the process*

$$\widetilde{S}_1(t) = S_1(0) e^{\alpha_1 t + |\sigma_1| W_1(t)} \tag{5.3a}$$

$$\widetilde{S}_2(t) = S_2(0) e^{\alpha_2 t + |\sigma_2| (\rho W_1(t) + \sqrt{1-\rho^2} W_2(t))}, \tag{5.3b}$$

where $|\sigma_i| = \sqrt{\sigma_{i1}^2 + \sigma_{i2}^2}$ *is the Euclidean norm of the vector* σ_i *and*

$$\rho = \frac{\sigma_1 \cdot \sigma_2}{|\sigma_1||\sigma_2|} \in [-1, 1] \tag{5.4}$$

is the cosine of the angle between σ_1, σ_2.

Remark 5.1. It can be shown the historical variances of the two stocks are unbiased estimators for $|\sigma_1|^2, |\sigma_2|^2$, while the historical correlation of log-returns of the two stocks is an unbiased estimator for ρ.

Now, exactly as in the one-dimensional case discussed in Section 0.6, it is possible to define a risk-neutral (or martingale) probability measure with respect to which the discounted value of both stocks are martingales. We seek this probability measure in the class of Girsanov probabilities $\mathbb{P}_\theta$ introduced in Theorem 0.15, where $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. The problem is solved in the following two theorems, which are the 2-dimensional generalizations of Theorem 0.18 and Theorem 0.19 respectively.

Theorem 5.2. *Let* $\mu = (\alpha_1 - r + \frac{1}{2}|\sigma_1|^2, \alpha_2 - r + \frac{1}{2}|\sigma_1|^2)$. *The discounted values* $S_1^*(t) = e^{-rt}S_1(t)$ *and* $S_2^*(t) = e^{-rt}S_2(t)$ *of the stocks have constant expectation in the Girsanov probability* $\mathbb{P}_\theta$ *if and only if* $\theta = q = (q_1, q_2)$, *where* q *is the (unique) solution of the linear system* $\sigma q = \mu$.

Theorem 5.3. *The stochastic processes* $\{S_1^*(t)\}_{t\geq 0}$ *and* $\{S_2^*(t)\}_{t\geq 0}$ *are martingales in the probability measure* $\mathbb{P}_\theta$ *if and only if* $\theta = q = (q_1, q_2)$.

Task 5.2. *Prove Theorem 5.2.*

The probability measure $\mathbb{P}_q$ is the martingale (or risk-neutral) probability of the 2-dimensional Black-Scholes market.

Task 5.3. *Prove that in the probability* $\mathbb{P}_q$ *the stocks prices are given by the following 2-dimensional geometric Brownian motion*

$$S_1(t) = S_1(0)e^{(r-\frac{|\sigma_1|^2}{2})t+\sigma_1\cdot W^{(q)}(t)}, \tag{5.5a}$$

$$S_2(t) = S_2(0)e^{(r-\frac{|\sigma_2|^2}{2})t+\sigma_2\cdot W^{(q)}(t)}, \tag{5.5b}$$

where $W^{(q)}(t) = (W_1^{(q)}(t), W_2^{(q)}(t)) = (W_1(t)+q_1t, W_2(t)+q_2t)$ *and* $\{W_1^{(q)}(t)\}_{t\geq 0}, \{W_1^{(q)}(t)\}_{t\geq 0}$ *are* $\mathbb{P}_q$*-independent Brownian motions. What is the equivalent (in distribution) expression for the process* (5.3)*?*

Denoting by $\mathbb{E}_q$ the expectation in the probability $\mathbb{P}_q$, the Black-Scholes price at time $t = 0$ of the 2-assets European derivative with pay-off Y at maturity T is given by the risk-neutral pricing formula

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y]. \tag{5.6}$$

Task 5.4. *Show that the risk-neutral price* (5.6) *of the relative outperformance option is given by*

$$\Pi_Y(0) = e^{(\hat{r}-r)T}\left(\frac{S_1(0)}{S_2(0)}\Phi(d_+) - Ke^{-\hat{r}T}\Phi(d_-)\right) \tag{5.7}$$

where

$$d_\pm = \frac{\log\frac{S_1(t)}{KS_2(t)} + (\hat{r} \pm \frac{|\sigma_1-\sigma_2|^2}{2})\tau}{|\sigma_1 - \sigma_2|\sqrt{\tau}}, \quad \hat{r} = \frac{|\sigma_1 - \sigma_2|^2}{2} + \left(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2}\right).$$

HINT: You need Theorem 0.10.

Task 5.5 (Matlab)**.** *Write a Matlab function*

`MaximumCall(K_1, K_2, T, s1, s2, sigma1, sigma2, rho, r, N, n)`

that applies the crude Monte Carlo method to compute the initial price (at time $t = 0$*) of the maximum call option with strikes* K_1, K_2 *and expiring at time* T. *Here* `s1`, `s2` *are the initial prices of the stocks,* `sigma1`, `sigma2` *are the volatilities of the stocks and* `rho` *is their correlation (in the notation* (5.3), `sigma1`$= |\sigma_1|$*). Moreover* N *is the number of points in a uniform partition of* $[0, T]$ *and* n *is the number of paths used for the Monte Carlo simulation. Plot the price and the confidence interval as a function of the number of paths. Plot also the price as function of the parameters* $K_1, K_2, s_1, s_2, \sigma_1, \sigma_2$ *and* ρ. *Discuss your findings.*

Integral form of the Black-Scholes price for general standard European derivatives on two stocks

In the case of standard European derivatives the risk-neutral pricing formula $\Pi_Y(0) = e^{-rT}\mathbb{E}[g(S_1(T), S_2(T)]$ can be written in a closed integral form, as shown in the following analog of Theorem 0.20.

Theorem 5.4. *The Black-Scholes price at time $t = 0$ of the 2-stocks option with pay-off $Y = g(S_1(T), S_2(T))$ is given by*

$$\Pi_Y(0) = v_0(S_1(0), S_2(0)), \tag{5.8a}$$

where the pricing function v_0 is given by

$$v_0(x, y) = \int_{\mathbb{R}^2} g\left(xe^{(r-\frac{|\sigma_1|^2}{2})T+\sqrt{T}\xi}, ye^{(r-\frac{|\sigma_2|^2}{2})T+\sqrt{T}\eta}\right) \frac{\exp\left(-\frac{1}{2}\begin{pmatrix}\xi & \eta\end{pmatrix}(\sigma\sigma^T)^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)}{2\pi\sqrt{\det(\sigma\sigma^T)}} d\xi\, d\eta. \tag{5.8b}$$

Sketch of the proof. The proof follows by using the joint density of $S_1(T), S_2(T)$ in the risk-neutral probability to compute $\mathbb{E}_q[Y]$. Namely, according to (5.5), the stock prices in the risk-neutral probability are given by a geometric Brownian motion with mean of log-returns $\alpha_j = r - \frac{1}{2}|\sigma_j^2|$, $j = 1, 2$. Replacing these values of α_1, α_2 into (5.2) gives the joint density $\widetilde{f}_{S_1(t),S_2(t)}(x, y)$ of the stock prices in the risk-neutral probability, from which we can compute $\Pi_Y(0) = e^{-rT}\mathbb{E}_q[g(S_1(T), S_2(T)]$ as

$$\Pi_Y(0) = e^{-rT}\int_{\mathbb{R}^2} g(x, y)\widetilde{f}_{S_1(T),S_2(T)}(x, y)\, dx\, dy.$$

After a proper change of variable, the previous expression transforms into (5.8). □

The Black-Scholes price at time $t > 0$ of the 2-assets European derivative with pay-off $Y = g(S_1(T), S_2(T))$ is given by a formula similar to the one in Theorem 5.4, namely

$$\Pi_Y(t) = v(t, S_1(t), S_2(t)),$$

where the pricing function v is given by

$$v(t, x, y) = \int_{\mathbb{R}^2} g\left(xe^{(r-\frac{|\sigma_1|^2}{2})\tau+\sqrt{\tau}\xi}, ye^{(r-\frac{|\sigma_2|^2}{2})\tau+\sqrt{\tau}\eta}\right) \frac{\exp\left(-\frac{1}{2}\begin{pmatrix}\xi & \eta\end{pmatrix}(\sigma\sigma^T)^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)}{2\pi\sqrt{\det(\sigma\sigma^T)}} d\xi\, d\eta.$$

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