MVE172 Basic Stochastic Processes and Financial Applications

Written re-exam Monday 11 April 2022 8.30-11.30

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) <u>or</u> Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Show that if for a discrete time Markov chain state i is recurrent and does not communicate with state j in sense of Hsu book (i.e., $i \not\leftrightarrow j$), then $p_{ij} = 0$. **(5 points)**

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a Wiener process with drift coefficient $\mu \in \mathbb{R}$ and $\mathbf{Var}\{X(1)\} = \sigma^2 > 0$. Find the conditional probability density function of $X(t)$ given that $X(s) = x$ for $0 < s < t$. **(5 points)**

Task 3. Let $X_1, X_2, \ldots$ be independent random variables with possible values $\{-1, 1\}$ and $\mathbf{P}\{X_i = -1\} = q = 1-p$ and $\mathbf{P}\{X_i = 1\} = p$. Show that $\{Y_n\}_{n=0}^{\infty}$ given by $Y_n = (q/p)^{X_1+\ldots+X_n}$ for $n \geq 1$ and $Y_0 = 1$ is a martingale. **(5 points)**

Task 4. Consider a taxi station where taxis and customers arrive in accordance with independent Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer that does not find a taxi waiting leaves. Find (a) the average number of taxis waiting, and (b) the proportion of arriving customers that get taxis. **(5 points)**

MVE172 Solutions to written re-exam 11 April 2022

Task 1. If $p_{ij} > 0$, then $p_{ji}(n) = 0$ for all n as otherwise i and j would communicate. But then the process starting in i has a probability at least $p_{ij} > 0$ of never returning to i which contradicts the recurrence of i.

Task 2. We may write $X(t) = \sigma\, W(t) + \mu\, t$ where $W(t)$ is a standard Wiener process so that $(X(t)|X(s) = x) = (\sigma W(t) + \mu t|\sigma W(s) + \mu s = x) = \sigma\,(W(t) - W(s)) + \mu\, t + (\sigma W(s)|\sigma W(s) = x - \mu s)$ so that $(X(t)|X(s) = x)$ is $\mathrm{N}(x + \mu(t-s), \sigma^2(t-s))$-distributed as $W(t) - W(s)$ and $W(s)$ are independent.

Task 3. This is Task 5.105 i the Hsu book which is a solved exercise on the course web page.

Task 4. The number of taxis waiting $\{X(t)\}_{t\geq 0}$ is a M/M/1 queueing system with $\lambda = 1$ and $\mu = 2$. Therefore (a) $\mathbf{E}\{X(t)\} = \lambda/(\mu - \lambda) = 1/2$ and (b) the proportion of arrival of customers that find at least one taxi waiting $1 - p_0 = \lambda/\mu = 1/2$.