MVE172 Basic Stochastic Processes and Financial Applications

Written exam Saturday 3 December 2022 8.30-11.30 AM

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $\{X(t)\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$. Find a (non-random) function $f:[0,\infty)\rightarrow\mathbb{R}$ such that $\{f(t)\mathrm{e}^{X(t)}\}_{t\geq 0}$ is a martingale wrt. the history of the X-process $F_s = \sigma(X(r): r\in[0,s])$. **(5 points)**

Task 2. Calculate the variance of $\int_0^1 X(s)\,ds$ for $\{X(t)\}_{t\geq 0}$ a Wiener process with $\mathbf{E}\{X(1)^2\} = 1$. **(5 points)**

Task 3. Let $\{X(n)\}_{n=0}^{\infty}$ be a discrete time Markov chain with state space $\{0,1,2\}$, starting value $X(0)=0$ and transition probability matrix $P = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$. Find the expected value E_{02} of the time $T = \min\{n>0: X(n)=2\}$ it takes the chain to reach the state 2. **(5 points)**

Task 4. Write a computer programme that by means of stochastic simulation finds an approximation of the expected value of the time it takes a M/M/1/2 queueing system with $\lambda=\mu=1$ starting empty $X(0)=0$ at time zero to become full (that is, server buzy and queuing slot occupied). **(5 points)**

MVE172 Solutions to written exam 3 December 2022

Task 1. $\mathbf{E}\{e^{X(t)}|F_s\} = e^{X(s)}\mathbf{E}\{e^{X(t)-X(s)}\} = e^{X(s)}\sum_{k=0}^{\infty} e^k \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)} = e^{X(s)} \times e^{(e-1)\lambda(t-s)}$ giving $f(t) = e^{-(e-1)\lambda t}$.

Task 2. As $\mathbf{E}\{\int_0^1 X(s)\,ds\} = 0$ we have $\mathbf{Var}\{\int_0^1 X(s)\,ds\} = \mathbf{E}\{(\int_0^1 X(r)\,dr)\,(\int_0^1 X(s)\,ds)\} = \int_0^1\int_0^1 R_X(r,s)\,drds = \int_0^1\int_0^1 \min(r,s)\,drds = 2\int_{s=0}^{s=1}\int_{r=0}^{r=s} r\,drds = 1/3$.

Task 3. Writing E_{12} for the expected value of the time it takes the chain to reach the state 2 starting at the state 1 we have the equations $E_{02} = 1 + (1/4)\,E_{02} + (1/2)\,E_{12}$ and $E_{12} = 1 + (1/4)\,E_{02} + (1/4)\,E_{12}$ with solution $E_{02} = 20/7$.

Task 4.

```
Repetitions = 1000000;
For[i = 1; Totaltime = 0, i <= Repetitions, i++,
  Time = Random[ExponentialDistribution[1]]; X = 1;
  While[X < 2, If[X == 1,
    Time = Time + Random[ExponentialDistribution[2]];
    If[RandomReal[{0, 1}] < 1/2, X = X - 1, X = X + 1],
      Time = Time + Random[ExponentialDistribution[1]]; X = 1]];
  Totaltime = Totaltime + Time];
Totaltime/Repetitions
```