MVE172 Basic Stochastic Processes and Financial Applications

Written exam Monday 3 April 2023 8.30-11.30 AM

TEACHER AND TELEPHONE JOUR: Patrik Albin 031 7723512.

AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) <u>or</u> Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate $\mathbf{P}\{X(1)=1\,|\,X(0)=0, X(2)=2\}$ for a Poisson process $\{X(t)\}_{t\geq 0}$.

(5 points)

Task 2. Calculate $\mathbf{P}\{X(1)\geq 1\,|\,X(0)=0\}$ for a zero-mean WSS Gaussian process $\{X(t)\}_{t\in\mathbb{R}}$ with autocorrelation function $R_X(\tau)=\mathrm{e}^{-|\tau|}$. [HINT: Use either that $X(t)-R_X(t-s)X(s)$ and $X(s)$ are uncorrelated or that $(X(s),X(t))$ has PDF $f_{X(s),X(t)}(x,y) = \frac{1}{2\pi\sqrt{1-R_X(t-s)^2}}\exp\{-\frac{x^2+y^2-2R_X(t-s)xy}{2(1-R_X(t-s)^2)}\}$.] **(5 points)**

Task 3. Find the probability mass function for the time T it takes for a discrete time Markov chain with state space $\{0,1,2\}$, starting at state 0 and with all transition probabilities equal 1/3 to reach the state 2. **(5 points)**

Task 4. Find a discrete time process $\{X_n\}_{n=0}^{+\infty}$ that is neither a martingale, a submartingale or a supermartingale wrt. $F_n=\sigma(X_1,\ldots,X_n)$. **(5 points)**

MVE172 Solutions to exam 3 April 2023

Task 1. $\mathbf{P}\{X(1)=1\,|\,X(0)=0,X(2)=2\} = \frac{\mathbf{P}\{X(0)=0,X(1)=1,X(2)=2\}}{\mathbf{P}\{X(0)=0,X(2)=2\}}$
$= \frac{\mathbf{P}\{X(0)=0,X(1)-X(0)=1,X(2)-X(1)=1\}}{\mathbf{P}\{X(0)=0,X(2)-X(0)=2\}} = \frac{\mathbf{P}\{X(0)=0\}\,\mathbf{P}\{X(1)-X(0)=1\}\,\mathbf{P}\{X(2)-X(1)=1\}}{\mathbf{P}\{X(0)=0\}\,\mathbf{P}\{X(2)-X(0)=2\}}$
$= \frac{\mathbf{P}\{X(0)=0\}\,\mathbf{P}\{X(1)=1\}\,\mathbf{P}\{X(1)=1\}}{\mathbf{P}\{X(0)=0\}\,\mathbf{P}\{X(2)=2\}} = \frac{1\cdot((1\cdot\lambda)^1\mathrm{e}^{-1\cdot\lambda}/(1!))^2}{1\cdot((2\cdot\lambda)^2\mathrm{e}^{-2\cdot\lambda}/(2!))} = \frac{1}{2}.$

Task 2. First solution: As $X(1)-\mathrm{e}^{-1}X(0)$ and $X(0)$ are uncorrelated we have $\mathbf{P}\{X(1)\geq 1\,|\,X(0)=0\} = \mathbf{P}\{X(1)-\mathrm{e}^{-1}X(0)\geq 1-\mathrm{e}^{-1}\cdot 0\,|\,X(0)=0\} = \mathbf{P}\{X(1)-\mathrm{e}^{-1}X(0) \geq 1\} = 1-\Phi\big(\frac{1-0}{\sqrt{\mathbf{Var}\{X(1)-\mathrm{e}^{-1}X(0)\}}}\big) = 1-\Phi\big(\frac{1}{\sqrt{1-\mathrm{e}^{-2}}}\big).$

Second solution: $\mathbf{P}\{X(1)\geq 1\,|\,X(0)=0\} = \int_1^\infty f_{X(1)|X(0)}(y|0)\,dy = \int_1^\infty \frac{f_{X(0),X(1)}(0,y)}{f_{X(0)}(0)}\,dy$
$= \int_1^\infty \frac{1}{2\pi\sqrt{1-R_X(1)^2}}\exp\big\{-\frac{y^2}{2(1-R_X(1)^2)}\big\}\big/\frac{1}{\sqrt{2\pi}}\,dy = \mathbf{P}\{\mathrm{N}(0,1-R_X(1)^2)\geq 1\} =$ as above.

Task 3. When the chain is in state 0 or 1 the probability that next state is still 0 or 1 is 2/3 while the probability that next state is 2 is 1/3. Hence the probability mass function is that of a waiting time distribution with $p=1/3$, i.e., $\mathbf{P}\{T=k\}=(1/3)\,(2/3)^{k-1}$.

Task 4. Let $X_0=0$ and $X_n=\sum_{k=1}^n Y_k$ for $n\geq 1$, where $\{Y_k\}_{k=1}^{+\infty}$ are independent random variables with $\mathbf{E}\{Y_k\}=1$ for k odd and $\mathbf{E}\{Y_k\}=-1$ for k even. Then $\mathbf{E}\{X_k|F_{k-1}\}=X_{k-1}+\mathbf{E}\{X_k\}$ which is bigger than X_{k-1} for k odd and smaller than X_{k-1} for k even.