MVE172 Basic Stochastic Processes and Financial Applications

Written exam Tuesday 22 August 2023 2-5 PM

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 8, 12 and 16 points for grades 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $X(t)$ and $Y(t)$, $t \geq 0$, be independent Poission processes with intensity (/rate) 1. Find an expression for $\mathbf{P}\{X(1) \geq Y(2)\}$ that can be readily calculated numerically using, for example, Mathematica or Matlab. **(5 points)**

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a zero-mean random process with autocorrelation function $R_X(s,t) = \min(s,t)$. Calculate $\mathbf{E}\{Y^2\}$ for $Y = X(0) + \int_0^1 X(s)\,ds$. **(5 points)**

Task 3. Consider a Markov chain $\{X_n\}_{n=0}^{+\infty}$ with state space $\{0,1,2\}$, initial probabil -ity distribution $\pi(0) = (1/2\ \ 1/2\ \ 0) = (1/2, 1/2, 0)$ and transition probability matrix P. Express the probability $\mathbf{P}\{X(1) = 2 \,|\, X(2) = 1\}$ in terms of the elements of P.

(5 points)

Task 4. Let $X_1, X_2, \ldots$ be independent random variables such that each X_i can take only two values $1/2$ and $3/2$ with probabilities $p \in [0,1]$ and $1-p$, respectively. For which values of p is $M_n = \prod_{i=1}^n X_i = X_1 \cdot \ldots \cdot X_n$ for $n \geq 1$, $M_0 = 1$, a submartingale?

(5 points)

MVE172 Solutions to exam 22 August 2023

Task 1. $\mathbf{P}\{X(1) \geq Y(2)\} = \sum_{k=0}^{\infty} \mathbf{P}\{X(1) = k\}\big(\sum_{\ell=0}^{k} \mathbf{P}\{Y(2) = \ell\}\big) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1^k}{k!}$ $\mathrm{e}^{-1} \frac{2^\ell}{\ell!} \mathrm{e}^{-2}$.

Task 2. As $\mathbf{E}\{X(0)^2\} = R_X(0,0) = \min(0,0) = 0$ we have $X(0) = 0$ giving that $Y = \int_0^1 X(s)\,ds$. Therefore we conclude that $\mathbf{E}\{Y^2\} = \mathbf{E}\big\{(\int_0^1 X(s)\,ds)(\int_0^1 X(t)\,dt)\big\} = \int_0^1 \int_0^1 \mathbf{E}\{X(s)X(t)\}\,dsdt = \int_0^1 \int_0^1 \min(s,t)\,dsdt = 2\int_{t=0}^{t=1} (\int_{s=0}^{s=t} s\,ds)\,dt = 1/3$.

Task 3. $\mathbf{P}\{X(1) = 2 \,|\, X(2) = 1\} = \frac{\mathbf{P}\{X(1)=2,\, X(2)=1\}}{\mathbf{P}\{X(2)=1\}} = \frac{\mathbf{P}\{X(2)=1 \,|\, X(1)=2\}\,\mathbf{P}\{X(1)=2\}}{\mathbf{P}\{X(2)=1\}} = \frac{P_{21}\,((1/2)\cdot P_{02}+(1/2)\cdot P_{12})}{(1/2)\cdot(P^2)_{01}+(1/2)\cdot(P^2)_{11}}$.

Task 4. With $F_n = \sigma(M_0,\ldots,M_n)$ we have $\mathbf{E}\{M_{n+1}|F_n\} = \mathbf{E}\{X_{n+1}M_n|F_n\} = M_n\,\mathbf{E}\{X_{n+1}\} = M_n\,((1/2)\cdot p + (3/2)\cdot(1-p)) = ((3/2)-p)\,M_n$ which is greater than or equal to M_n making M_n a submartingale if and only if $p \leq 1/2$.