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Cite as: AIP Conference Proceedings **2293**, 040003 (2020); https://doi.org/10.1063/5.0026917
Published Online: 25 November 2020

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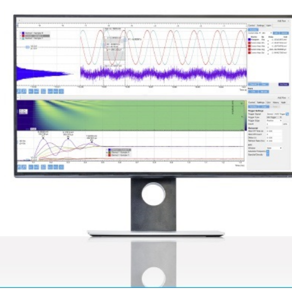

The Relevant Role of Calculus in Renaissance Domes' Design, Before Differential Geometry was Born

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Abstract. We take under consideration two masonry domes built during Italian Renaissance: the Santa Maria del Fiore dome in Florence by F. Brunelleschi and St. Peter's dome in Rome by Michelangelo. No original computation is known currently in reference to neither one nor the other. However present mathematical computations show that, even though probably unconsciously, the designs of both the domes were inspired by the use of catenary, a trascendent mathematical curve with excellent stability properties. It was defined and studied in details by Leibniz and Bernoulli brothers at the end of the 17^{th} century (within the context of rising differential geometry) a long time later than both the domes were completely built. Therefore the two considered domes suggest that some practical applications of differential geometry in architecture design can be considered foregoing its theoretical formalization. Here we show some numerical computation which support this hypothesis. The aim of this work is to show that knowledge quite often is created in processes and not invented in a moment; in particular here we focus on the development of differential calculus related to its applications to equilibrium of masonry domes.
Keywords: catenary, funicular surface, masonry domes, Renaissance domes, mathematical knowledge development

INTRODUCTION

In Architecture design, the dome reveals a prominent element along the centuries and in many cultural contexts, in particular when great churches and public buildings with symbolic relevance are involved.

From the mathematical point of view, a dome can be well approximated by a rotation solid whose cross-section provides the generating curve. Obviously, a frequent generating curve is the semicircumference (such as in the Pantheon dome in Rome), but here we focus on other curves; in particular we consider parabola curve, which was known since the 4^{th} century B.C. by the work of Greek mathematician Menecmus, and **catenary curve** (in this case a so called funicular surface is obtained), which was studied and introduced just at the end of 17^{th} century, after that the differential geometry was developed. In that period the general differential calculus was born, which allowed to understand the significant stability properties of catenary with respect to parabola. Nevertheless, a catenary curve can be detected running within the thickness of the walls of Brunelleschi's dome in Florence (completed in 1436) and in St. Peter's dome in Rome (completed in 1590 by a disciple of Michelangelo, who followed the original design because of Michelangelo's death).

This shows how good were ancient builders who tried to fit the catenary curve within the wall thickness of their domes either by an intuitive insight or simply by a trial and error method, since a conscious design would imply knowledge of differential calculus which was not yet formalized in those centuries.

This opens the question of what can be considered mathematical knowledge. Actually, the historical evidence shows that knowledge and principles required to build historical architecture were available much before that such knowledge was mathematically assessed and formalized. However, we do not conclude that mathematical formalization always acts as a second step. Instead, we bring to the reader's attention some facts to show that *knowledge does not follow linear paths*.

International Conference of Numerical Analysis and Applied Mathematics ICNAAM 2019
AIP Conf. Proc. 2293, 040003-1–040003-4; https://doi.org/10.1063/5.0026917
Published by AIP Publishing. 978-0-7354-4025-8/$30.00

CATENARY vs. PARABOLA

The parabola curve is the graphic of any analytical function which is a polynomial of degree two. From the point of view of classical geometry, any parabola can be built as a conic section, Unfortunately, parabola has no stability property. Instead, catenary has excellent stability properties, but its equation is quite more complicated, involving a hyperbolic cosine, which was unknown in the 15^{th} and 16^{th} centuries. Indeed, catenary is the curve which represents the shape of a hanging chain (or an inextensible cable) of uniform mass per unit of length, fixed at the ends and subject only to his own weight. Such curve is given by

$$y = a\cosh\left(\frac{x - x_0}{a}\right) + (y_o - a)$$

where (x_0, y_0) is the lowest point on the curve. If w is the weight per unit length and T_0 is the cable tension at the lowest point, then

$$a = T_0/w$$

is the so called *catenary constant*. For increasing values of a, catenary exhibits increasing spans.

If catenary is "frozen" and inverted, the chain (or cable) exhibits the shape of an upside-down catenary, subject to a compression strength only, which means that it supports itself [see Figure 1]. This fact was discovered by Robert Hooke in 1671, when he announced at the Royal Society in London he had found the shape of the optimal arch. He published this result in 1676 in a book where he states: *Ut continuum pendet flexile, sic stabit continuum rigidum inversum* (as a flexible cable hangs, so, inverting it, a rigid body stands still) [1]. By the way, these ideas were very useful to architect Christopher Wren who in those years accepted to rebuild St. Paul's Cathedral in London [1] and was surely inspired by Hook's intuition. Hooke, who was essentially an engineer, never provided the proof, since he was unable to study in deep the mathematical analysis about the catenary curve. However, in the same years, at the end of the 17^{th} century, catenary was widely studied also by famed mathematicians such as Leibniz, Huygens and the brothers Johannes and Jacob Bernoulli, who competitively and successfully provided the analytical solution of the differential equation for catenary equilibrium, adding rich theoretical details. About that, it is worth citing [2], which is a history of Mathematics, written in 1758 just a few years after the birth of differential calculus and now available in digitalized form. Moreover, the reader can be interested in a contemporary comment to that work in modern notations [3], which is devoted to the original formal development of catenary.

The first conscious example of catenary in Architecture refers to St. Peter's dome in Rome.

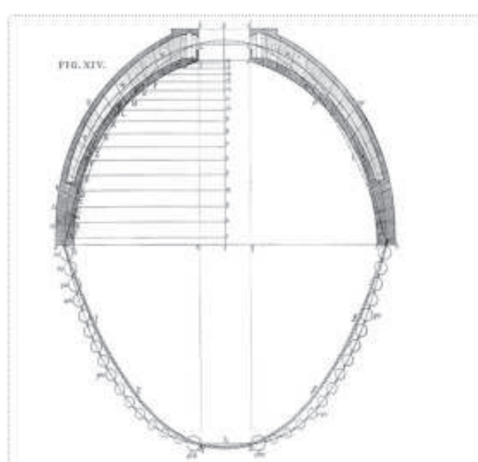

FIGURE 1. Original drawing of St. Peter's dome by G. Poleni [4]

As well as Brunelleschi, Michelangelo did not write anything about his project and this was a great disadvantage when by the end of the 17^{th} century (about a century after that it was completed) the dome started to show a serious chance of collapsing. In 1743 the Pope assigned to Giovanni Poleni the task of studying the structure and solving the problem. Poleni was a renowned mathematician and engineer and consequently he knew very well the role of catenary in structure stability; therefore he built small models in scale of the catenary running between the two shells of St. Peter's dome and wrote down his computations. Figure 1 reports one of his original drawings, where catenary is easily recognizable in both its classical shape and the inverted shape [4]. Poleni concluded that the shape of the Michelangelo's dome was satisfactory. So the structure was simply strengthened and even now we can admire the efficiency of that action based on the use of catenary as a mathematical model [5].

Therefore we can conjecture that Michelangelo himself was aware of the stability properties of catenary which obviously he did not know as an analytical curve, but probably as a particularly convenient parabola. Indeed any catenary can be well approximated by convenient parabolas. Analogously, we can conjecture that even Brunelleschi knew convenient parabolas which guaranteed better stability to his dome, which still stands firmly nowadays after just minor stability problems, during the last five centuries. Michelangelo stated explicitly that he was inspired by Brunelleschi, so our conjecture is well founded.

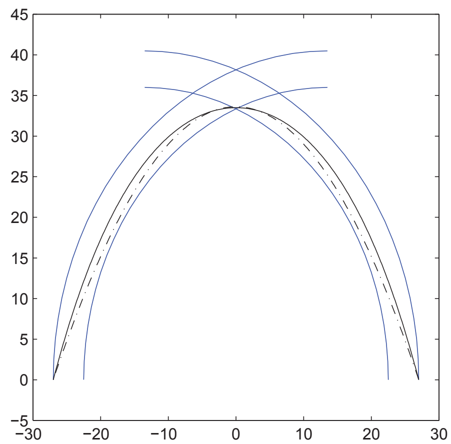

FIGURE 2. Computed catenary and related parabola within the wall thickness of Brunelleschi's dome

Actually the Brunelleschi's dome is made up by eight membranes, based on an octagon; however it can be well approximated by a rotation solid. During the last centuries many theories were proposed about the technique used by Brunelleschi to build the dome. One of the most reliable theories was presented by L. Ximenes in 1757 [6]. According to this theory, between the two shells of the dome, actually an idealized inverted catenary curve runs. The computation and drawing of this curve were presented by G. Conti [7].

But the question arises: is that catenary an occasional find?

Indeed, the shape of the dome is what is called "pointed fifth" for the inner shell and "pointed fourth" for the outer shell. Between such kinds of two shells, **we found that it is always possible to make a catenary run and a conveniently close parabola as well.** Obviously, such parabola inherits the proportion of catenary and its stability properties. Figure 2 shows an example of such curves between the two shells using real measures of Brunelleschi's dome (basis diameter equal to 54 m and maximum height equal to 33.5 m): in solid line the catenary within the wall thickness and in dashed line the parabola fitting the given catenary with an error of 3%. Computations, evaluations and figure are by the author. Remarkably, parabola exhibits the same endpoints and the same maximum as related catenary. Brunelleschi was able to build parabolas by tools available in classical geometry. Probably, he chose among all the possible parabolas that one which runs between the pointed fifth shell and the pointed fourth shell because he was aware of its stability properties.

Now we know that he was right because he intuitively used the catenary curve.

CONCLUSION

We can conclude citing [8] "Knowledge is created from a subjective combination of different attainments as there are intuition, experience, information, education, decision, power of persuasion.(...) Therefore, not all the knowledge can be explicitly formalized." Here we presented a historical example of applications of such process of knowledge, at first used in an unformalized way by architects and then formalized by mathematicians. When knowledge can be formalized it becomes heritage of all the mankind, but this seems to be the last step. Actually, we keep on investigating and studying because the target of knowledge can be always moved step by step a little further. The contribution of this paper is to show how this process was followed and successful even in Italian Renaissance masonry domes design.

ACKNOWLEDGMENTS

The author is grateful to Claus-Peter Rückemann for his kind helpfulness and comments.

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