The Brachistochrone Problem: An application of Calculus of Variations

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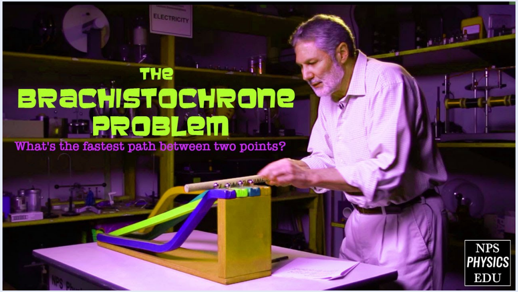
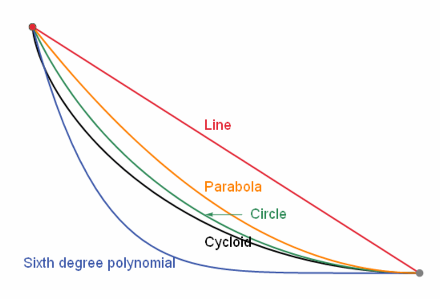


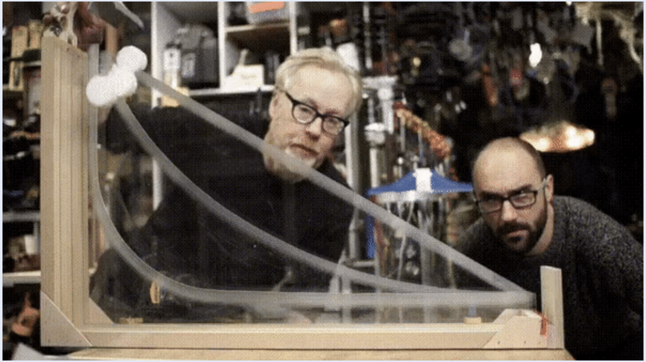
Calculus of variations: The Brachistochrone Problem

- Bernoulli (1696): "Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time." $\Rightarrow$ *Brachistochrone*[1] problem



[1] from Ancient Greek: brákhistos khrónos, i.e. 'shortest time'

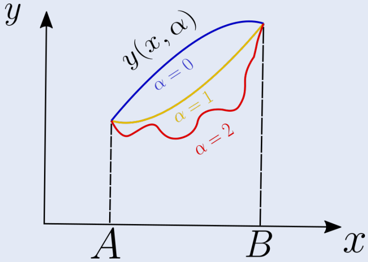
The Brachistochrone - Mythbusters experiment



- Problems such as the Brachistochrone problem can be solved using *calculus of variations* (Euler 1700s)
- Idea: finding the *function* $y(x,\alpha)$ which makes the integral I *stationary*:

$$\boxed{I(\alpha) = \int_A^B F(y(x,\alpha), y', x)dx \ , \quad \frac{dI}{d\alpha}|_{\alpha=\alpha_0=0} = 0} \tag{1}$$

- F is a functional ($\sim$ function of functions), $y(x,\alpha) = y(x, \alpha = 0) + \alpha f(x)$, $f(x)$ being a smooth deformation with $f(A) = f(B) = 0$



Calculus of variations: crash course cont.

- From the stationary condition $\frac{dI}{d\alpha} = 0$ the *Euler equation* follows (or the *Beltrami identity* if F not expl. dependent on x):

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \text{ (Euler)} \quad , F - y'\frac{\partial F}{\partial y'} = C \ \text{ (Beltrami id.)}} \tag{2}$$

- From math to physics: What can $I(\alpha)$ correspond to in the real world?

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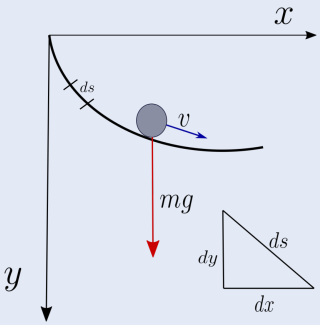
- From math to physics: What can $I(\alpha)$ correspond to in the real world?
- Energy, area, distance...., TIME!

Calculus of variations: applied to BP

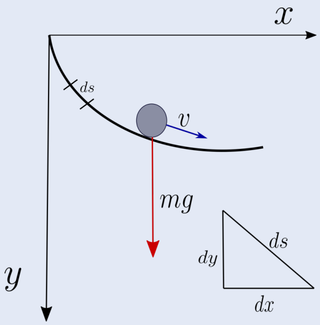


- Galileo:

$$t = \int \frac{ds}{v} \ , ds = \sqrt{dx^2 + dy^2} \tag{3}$$

- How to compute the speed v?

Calculus of variations: applied to BP



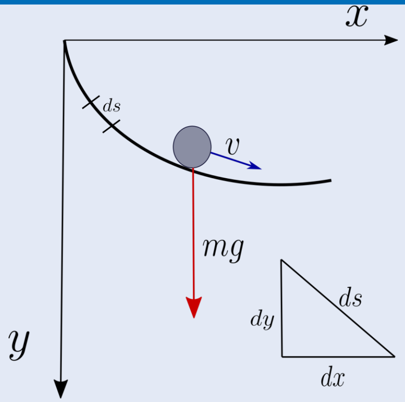
- Galileo:

$$t = \int \frac{ds}{v} \, , ds = \sqrt{dx^2 + dy^2} \tag{3}$$

- How to compute the speed v? Conservation of energy!

$$E_{\text{pot}} = E_{\text{kin}} \, , \Leftrightarrow mgy = \frac{mv^2}{2} \Rightarrow v = \sqrt{2gy} \tag{4}$$

Calculus of variations: applied to BP



- Combine the results on the previous slide:

$$t = \int F(y, y', x)dx \;, F(y, y', x) = \sqrt{\frac{1 + y'^2}{2gy}} \tag{5}$$

Now: find $y(x)$ such that t is *minimized*

Calculus of variations: applied to BP cont.

- Recall the Beltrami identity (since F not expl. dep. on x!):

$$F - y' \frac{\partial F}{\partial y'} = C \, , F(y, y') = \sqrt{\frac{1 + y'^2}{2gy}} \tag{6}$$

- Plug in F and perform the derivatives! Result can be rewritten according to (exercise...)

$$\frac{dy}{dx} = \sqrt{\frac{K - y}{y}} \, , K = \frac{1}{2gC^2} \Leftrightarrow \int dx = \int dy \sqrt{\frac{y}{K - y}} \tag{7}$$

[2] The curve traced out by a point on the rim of a wheel as the wheel rotates.

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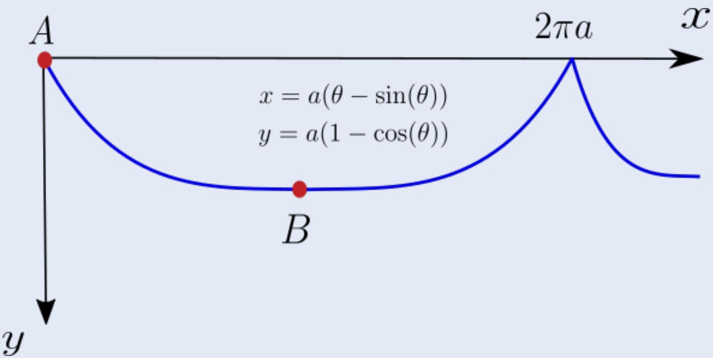
- Solve the y-integral by the substitution $y = K \sin^2(\theta/2)$
- The solution is described by an *inverted cycloid*:[2]

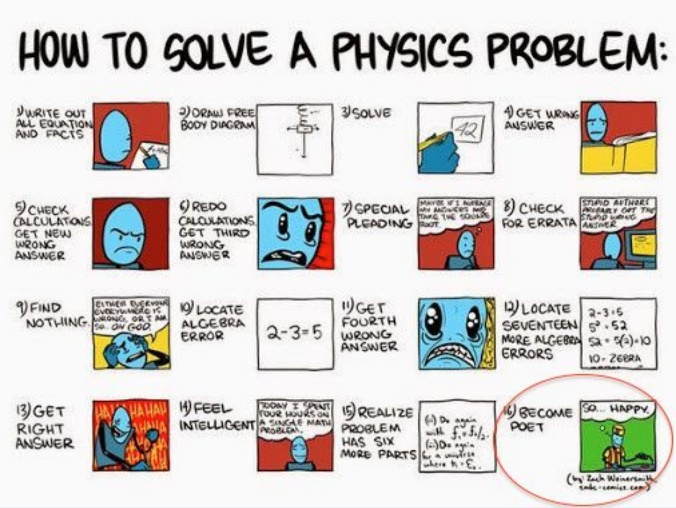
$$\boxed{x = \frac{K}{2}(\theta - \sin(\theta)) + D \, , y = \frac{K}{2}(1 - \cos(\theta))} \tag{8}$$

[2] The curve traced out by a point on the rim of a wheel as the wheel rotates.

Calculus of variations: applied to BP cont.



- Many other COOL applications of CoV: optimal shape of soap bubble, hanging rope, prove that the shortest distance between two points is a line, fencing the largest possible area (Didos problem) etc etc...


HOW TO SOLVE A PHYSICS PROBLEM:
1) WRITE OUT ALL EQUATIONS AND FACTS
2) DRAW FREE BODY DIAGRAM
3) SOLVE
42
4) GET WRONG ANSWER
5) CHECK CALCULATIONS GET NEW WRONG ANSWER
6) REDO CALCULATIONS GET THIRD WRONG ANSWER
7) SPECIAL PLEADING
MAYBE IF I AVERAGE MY ANSWERS AND TAKE THE SQUARE ROOT.
8) CHECK FOR ERRATA
STUPID AUTHORS PROBABLY GOT THE STUPID WRONG ANSWER.
9) FIND NOTHING.
EITHER EVERYONE EVERYWHERE IS WRONG, OR I AM. OH GOD.
10) LOCATE ALGEBRA ERROR
2-3=5
11) GET FOURTH WRONG ANSWER
12) LOCATE SEVENTEEN MORE ALGEBRA ERRORS
2-3=5
5²=52
52=5(2)=10
10=ZEBRA
13) GET RIGHT ANSWER
14) FEEL INTELLIGENT
TODAY I SPENT FOUR HOURS ON A SINGLE MATH PROBLEM.
15) REALIZE PROBLEM HAS SIX MORE PARTS
16) BECOME POET
SO... HAPPY.
(by: Zach Weinersmith, smbc-comics.com)

Additional details (derivation of Euler equation)

- Goal: make $I(\alpha)$ stationary

$$\frac{dI}{d\alpha} = \int_A^B \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)dx = 0 \tag{9}$$

- Introduce $\delta y = y(x, \alpha) - y(x, 0) = \alpha f(x)$ for some deformation $f(x)$ with $f(A) = f(B) = 0$. $y(x, 0)$ is assumed to solve the optimization problem.

$$\frac{\partial y}{\partial \alpha} = f(x)\,, \frac{\partial y'}{\partial \alpha} = \frac{\partial f}{\partial x}\,. \tag{10}$$

- Plug (10) into (9) and perform integration by parts to the second term (boundary terms vanish!)

$$\frac{dI}{d\alpha} = \int_A^B \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)f(x)dx = 0 \tag{11}$$

- $f(x)$ arbitrary $\Rightarrow$ Euler equation!