Appendix E

Group Theory

In this appendix we give an introduction to group theory as it is applied in physics. We will be covering the basic definition of a group, and provide some additional definitions. After this we give a brief introduction to representation theory, and then discuss a special class of groups called *Lie groups*. A Lie group is a group that is also a differential manifold, which is of fundamental importance in physics. To provide a more thorough understanding and some physical context, we also provide a cursory introduction to *group manifolds* and *spin representations*. The chapter is then concluded with explicit calculations of Lie algebras for use throughout the thesis.

Definition E.0.1. A group is defined as a set of unique elements ($\mathbf{G}$) which together with an operation ($*$) fulfills the following:

Closure : *if* a *and* $b \in \mathbf{G}$*, then* $a * b \in \mathbf{G}$.

Associativity : $\forall a, b$ *and* $c \in \mathbf{G}$*,* $(a * b) * c = a * (b * c)$.

Unit element : *there exists an object* $e \in \mathbf{G}$ *such that* $a * e = e * a = a$.

Inverse element : *for each* $a \in \mathbf{G}$*, there is an element* $b \in \mathbf{G}$ *such that* $a * b = b * a = e$.

The most common example of an easy to understand group is the set of all integers combined with the operation of addition. The sum of two integers is always an integer, and we already know that addition is associative. The unit element of the group is 0, since adding it to any integer will yield that integer as a result. Finally, the inverse element of any integer is just the same integer with opposite sign.

In physics we use the group structure to express certain symmetries of a system, such as invariance under a rotation, or Lorentz invariance. It can be shown that, with an appropriate operator, these symmetries can be expressed as groups and represented as matrices. Group theory can then be used to find the equations of motions for a system purely by using the symmetries that the theory possesses.

To make use of group theory there are some tools we need to understand and use, so we continue with a few more definitions:

Definition E.0.2. A group is called **abelian** if the elements commute with regards to the group operation ($*$), that is:

$$\forall a,b \in \mathbf{G} \ : \quad a * b = b * a \ .$$

This is important because the difference between what mathematical tools are applicable to abelian and non-abelian groups is very big. A prime example of a non-abelian group is the set of all invertible $n \times n$ matrices combined with the matrix multiplication operator. The difference is also readily observable in sections 2.2 and 3.

Definition E.0.3. If $\mathbf{G}$ and $\mathbf{H}$ are two groups with the same group operator ($*$), $\mathbf{H}$ is said to be a subgroup of $\mathbf{G}$ if

$$a \in \mathbf{H} \ \Rightarrow \ a \in \mathbf{G}$$

There are always two trivial subgroups; the unit element and the group itself.

Definition E.0.4. A map $\psi : \mathbf{G} \mapsto \mathbf{H}$ is a **homomorphism** if

$$g_i * g_j = g_k \ ,$$

and

$$\psi(g_i) * \psi(g_j) = \psi(g_k) \ .$$

where $g_i, g_j, g_k \in \mathbf{G}$, $\psi(g_i) * \psi(g_j), \psi(g_k) \in \mathbf{H}$. If ψ is also bijective it is an isomorphism, which we denote with $\mathbf{G} \cong \mathbf{H}$.

E.1 Representation theory

The most simple, but perhaps most instructive example of a continuous group is the group of all rotations in two dimensions. While we can write down a general element of the group of all rotations in two dimensions as $g(\theta)$, where θ is a continuous parameter, we would like to construct a more explicit representation. To do this we associate each element of the group with a matrix, taking the matrix multiplication as the group operator. If we fix a coordinate system we may express $g(\theta)$ with the rotations matrix $R(\theta)$ defined as

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} .$$

The idea of associating group elements with matrices is formally known as *representation theory.* In mathematics group theory and representation theory can certainly be considered different subjects but in physics a distinction is seldom made.

Now that we have introduced the idea of representing group elements with matrices we are ready to continue. While it was easy to write down the rotation matrix in two dimensions we would certainly like to be able to rotate things in more than one plane. Since our intuition as humans is for the most part only good in two to three dimensions we would like to find a general way of constructing rotation matrices in arbitrary dimension. Thus, we must ask ourselves what property defines a rotation. The common answer is that a rotation is a transformation that preserves the length of vectors. Thus we want to find all matrices that leaves the quantity $x^T x$ invariant. We consider an arbitrary transformation with a matrix M

$$x^T x \mapsto (Mx)^T(Mx) = x^T M^T Mx \ .$$

Now, if this quantity is to be equal to $x^T x$ we must have $M^T M = 1$. We call matrices that satisfies this condition *orthogonal.* But we are not done, if we take the determinant of the defining equation of a orthogonal matrix we find that $\det(M) = \pm 1$. Matrices which switch parity are orthogonal matrices with determinant -1. To exclude these we also demand that a rotation matrix M satisfies $\det(M) = 1$. These two demands are usually summarized by saying that a rotation is an element of the group $SO(n)$ where S stands for *special*, meaning that we have a determinant of one. O is for *orthogonal* and n is the dimension of the matrix and, obviously, of the space.

E.2 Lie Groups

In physics we are mostly interested in continuous groups. A simple example apart from rotations are the group of real numbers. Of special interest are the continuous groups known as *Lie groups* (named after the brilliant Norwegian mathematician Sophus Lie (1842-1899)), as they are differential manifolds meaning they describe some continuous geometry. This allows us to perform certain operations on the group manifolds, namely integration and differentiation. Lie groups are defined as follows:

Definition E.2.1. A Lie group ($\mathbf{G}$) is a finite-dimensional differential manifold with an associated smooth multiplication map:

$$(g_j, g_i) \in \mathbf{G} \times \mathbf{G} \rightarrow g_i g_j \in \mathbf{G}$$

and a smooth inverse map

$$g \in \mathbf{G} \rightarrow g^{-1} \in \mathbf{G}$$

that satisfy the group axioms in E.0.1.

When we construct the *covariant derivative* in section 2.1 there is a necessity for a theory of the first spatial derivative of the transformations corresponding to the Lie groups. We refer to these terms as being *Lie algebra-valued.* The theory regarding these objects is that of *Lie algebra.* The Lie algebra can be seen as a minimal representation of a group using group elements infinitely close to the identity element as *generators* that can span the entire group.

Definition E.2.2. A Lie algebra $\mathbf{g}$ is a vector space which has a bilinear mapping $[.,.] : \mathbf{g} \times \mathbf{g} \mapsto \mathbf{g}$ such that:

For all $X,Y \in \mathbf{g}$
$$[X,Y] = -[Y,X] \ .$$

The Jacobi identity,
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \ .$$

The objects X, Y, Z of this definition are usually denoted T_i where i is some index, and are called *generators*. The brackets([.,.]) are generally referred to as *Lie brackets*.

Now, how can we construct our original rotation matrix from these generators? To proceed we will have to use a idea, courtesy of Sophus Lie. Lie proposed that instead of performing the whole transformation at once, we can split it into many small transformations. Let $U(\phi)$ be a rotation transformation and, as discussed before, also a matrix. For a small transformation we may expand our transformation around the identity as

$$U(\delta\phi) = \mathbb{1} + \delta\phi T \ .$$

Now we can express a full transformation $U(\phi)$ by first dividing up the parameter in N pieces, $\phi = N\delta\phi$. To compensate we of course have to perform the transformation N times. We can now take the limit as N tends towards infinity of N infinitesimal transformations performed in succession:

$$U(\phi) = \lim_{N\to\infty} (\mathbf{1} + \frac{\phi T}{N})^N.$$

The infinity-limit on N defines the exponential, so we have

$$U(\phi) = \mathrm{e}^{\phi T} \ .$$

We see that to construct a full transformation we only need to find the generator and then exponentiate it. It is important that the exponential of a matrix should be interpreted as an infinite series according to
$$\mathrm{e}^X = \sum_{i=0}^{\infty} \frac{X^n}{n!} \ .$$

Let us return to our rotation matrix and find it again using the method described above. First we need to find the generator T. By expanding the defining equation for $SO(2)$ infinitesimal we find

$$(\mathbb{1} + i\delta\phi T)^T(\mathbb{1} + i\delta\phi T) = 1 \implies T^T + T = 0 \ ,$$

if the equation is to hold to first order. Now, there are not many 2×2 matrices which satisfy this condition. Truth is, down to a scale factor, c, there is only one:

$$T = c \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Now, using the exponential equation we find

$$e^{\phi cT} = \sum \frac{(\phi cT)^n}{n!} = \begin{bmatrix} \sum_{n=1} \frac{(-c)^{(n+1)}(\phi)^{2n}}{(2n)!} & \sum \frac{(-c)^{n+1}(\phi)^{2n-1}}{(2n-1)!} \\ -\sum \frac{(-c)^{n+1}(\phi)^{2n-1}}{(2n-1)!} & \sum \frac{(-c)^{n+1}(\phi)^{2n}}{2n!} \end{bmatrix}_{c=1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix},$$

where we in the last step see that we need to set $c = 1$ to get the rotation matrix out of our endeavour. We have managed to retrieve the classical rotation matrix, and now you may wonder for which groups are these manipulations actually allowed. It turns out that this procedure works for any group whose elements we may write as $U(\phi_1,\phi_2,..\phi_n)$ given that the parameters are continuous and that there exists $U(\phi_1,\phi_2,..\phi_n) = \mathbf{1}$ for some set of ϕ. This restriction is the same as saying that the parameters define a differentiable manifold, as a manifold consisting of a set of orthogonal, continuous parameters is trivially differentiable.

Now that we have confirmed that Lie's claim holds we can do this the other way around! Remembering that $U(\phi) = \mathrm{e}^{T\phi}$, differentiating both sides with respect to ϕ and then letting $\phi \to 0$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}\phi} U(\phi) = TU(\phi) \ , \tag{E.1}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\phi} U(\phi)\right]_{\phi=0} = T \cdot U(0) \ , \tag{E.2}$$

since U is a transformation around the unit element $U(0)$ is just the identity, giving us an expression for the generator T in terms of the parametrized representation matrix.

Now we have only discussed groups with one parameter, so we turn our attention to groups with multiple parameters. For groups with multiple generators we can further develop our understanding of them by attempting to combine them. We begin with expanding the group operator according to

$$U_a(\delta\phi) = \mathrm{e}^{(T_a \ \delta\phi_a)} = 1 + T_a \delta\phi_a + \frac{1}{2} T_a^2 (\delta\phi_a)^2 \ .$$

Using this notation we can combine two generators T_a and T_b:

$$U_b^{-1} U_a^{-1} U_b U_a = 1 + \delta\phi_a \delta\phi_b [T_a, T_b] + \ldots \ .$$

We can choose to ignore higher order terms because we have assumed $\delta\phi_a, \delta\phi_b$ to be small. Since the group is closed $[T_a, T_b]$ must either be a group generator or zero. We see from this that

$$[T_a, T_b] = f_{ab}{}^c T_c \ , \tag{E.3}$$

must hold. T_c denotes some arbitrary group generator for the same group as T_a, T_b. The constants, $f_{ab}{}^c$ are called the *structure constants*. It can be shown that $f_{ab}{}^c$ is antisymmetric in its three indices, although some care needs to be taken when exchanging raised and lowered indices. The structure constant provides a representation independent characterization of a Lie group. This is a good time to look again at the definition of a Lie algebra. The commutator is the bilinear operation in its definition and it fulfills the Jacobi identity.

Finally, we note that the sum of two elements in the Lie algebra are in the Lie algebra, since the product of two elements in the Lie group is in the Lie group, and multiplication corresponds to addition in the exponent.

E.2.1 Killing forms

So far we have introduced the Lie bracket of a Lie algebra so that given two elements in the algebra we may construct a third one. However, we may ask for another operator that given two group elements returns a scalar. What we are seeking is thus a generalisation of the scalar product from linear algebra. Before we define this operator we first introduce a new notation for the Lie bracket with

$$\mathrm{ad}_x(y) = [x, y] \ .$$

The operator $\mathrm{ad}_x(y)$ is called the *adjoint action*. The reason we introduce this notation is because the use of Lie brackets may become cumbersome. For example consider the expression

$$[x[x,[x,[x,y]]]] = \mathrm{ad}_x \circ \mathrm{ad}_x \circ \mathrm{ad}_x \circ \mathrm{ad}_x(y) = \mathrm{ad}_x^4(y) \ .$$

It is clear that the right hand side is a more appropriate notation. We are now ready to give a definition:

Definition E.2.3. Given a finite-dimensional Lie algebra g its **Killing form** is the symmetric bilinear form given by the formula

$$\kappa(x, y) = \mathrm{tr}[\mathrm{ad}_x \circ \mathrm{ad}_y] \ .$$

Remember that in our notation this reads $ad_x \circ ad_x = [x[y,\cdot]]$. The Killing form is therefore a matrix, given a basis. Of course we shouldn't be surprised, there is after all a trace in the definition. Now, let us consider the special case of the adjoint action of a generator. To emphasize we write $x = T^a$. Its action on another generator is by definition

$$\mathrm{ad}_{T^a}(T^b) = [T^a, T^b] = f^{ab}{}_c T^c \ ,$$

where f is the structure constant. Since all elements of a Lie algebra can be written as a linear combination of the generators we can use these as a basis. Construct a vector, x, where the a:th entry indicates the multiplicity of T^a. Then we may write down a matrix for the adjoint action as

$$(\mathrm{ad}_{T^a})_{bc} = f^{ab}{}_c \ .$$

Just applying the same idea we find

$$\mathrm{ad}_{T^a} \circ \mathrm{ad}_{T^b} = (\mathrm{ad}_{T^a})_{cd}(\mathrm{ad}_{T^b})_{de} = f^{ac}{}_d f^{bd}{}_e \ ,$$

and we thus reach

$$\mathrm{tr}[\mathrm{ad}_{T^a}\mathrm{ad}_{T^b}] = \mathrm{tr}[f^{ac}{}_d f^{bd}{}_e] = f^{ac}{}_d f^{bd}{}_c \ ,$$

where we have summation over c and e.

E.3 An introduction to group manifolds

The properties of the group manifold has some implications for the physics that come from that group. Because of this it is necessary to be at least somewhat familiar with group manifolds, so we introduce them here. In general, the group manifold is an isomorphism class of manifolds, rather than some particular manifold. One way of finding a manifold that is part of this isomorphism class for a matrix representation of a group is by finding the parameter space of the matrix representation. To build intuition we will go over a few examples and discuss them afterwards.

The purpose of this section is for the reader to gain some understanding for the group manifold, a general manifold and the *direct product*. Examples of direct products relevant throughout this thesis are $SL(2) \times SL(2)$ and $SL(3) \times SL(3)$, which are the basis of the theory of Chern-Simons gravity that we present.

The special unitary group of order 2 can be written on the following parametric form

$$\begin{bmatrix} a+ib & -c+id \\ c+id & a-ib \end{bmatrix} \ ,$$

with the restriction $a^2+b^2+c^2+d^2 = 1$ coming from the restriction on the determinant. It is easily seen that the group manifold of $SU(2)$ in this representation is a hypersphere. The hypersphere is isomorphic to the three dimensional ball by isomorphisms of the form

$$\begin{cases} x = a \ , \\ y = b \ , \\ z = c \ , \\ r^2 = 1 - d^2 \ . \end{cases}$$

We see this because all of a,b,c,d are already limited by values between 0 and 1, and we've transferred the equation to one of the form $x^2+y^2+z^2 = r^2$ where the radius can be between 0 and 1.

The group $SO(1{,}1)$ is just the group of all Lorentz tranformations in one direction. The transformation matrix can be written as

$$\begin{bmatrix} p & s \\ s & p \end{bmatrix}$$

where the orthogonality condition is taken care of by the Minkowski metric. The determinant restriction gives us the parameter restriction $p^2 - s^2 = 1$ which defines the unit hyperbola. The hyperbola is an obvious example of a *non-compact* Lie group, as the hyperbola continues off into infinity meaning there is no well-defined final point. It is also not fully connected, because the allowed values for p are $p \leq -1, p \geq 1$, leaving a hole in the middle.

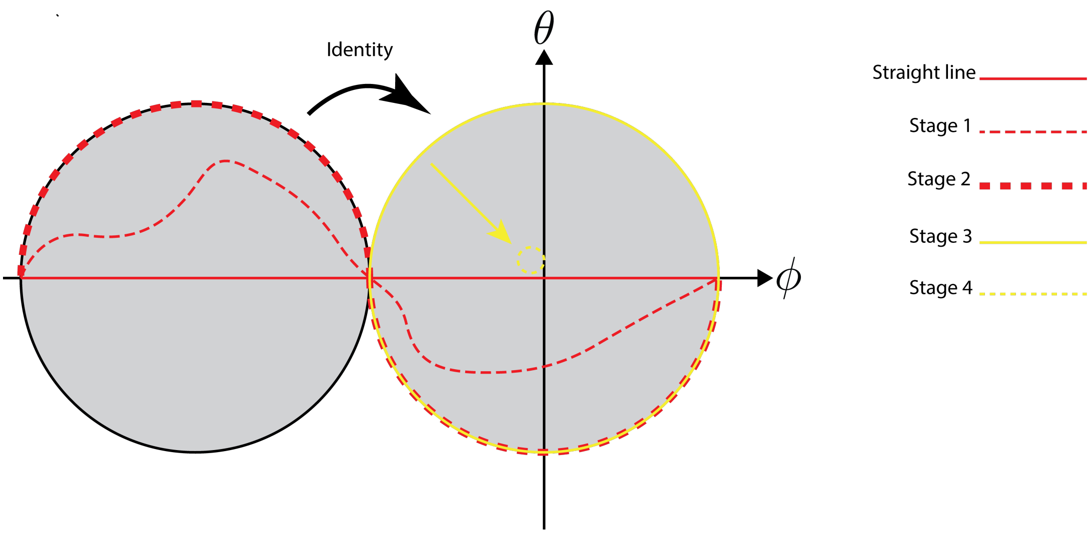
As we have seen in E.5.1 rotations in three dimensions can be represented as a multiplication of three rotations around different axes by some angle. Each of these matrices takes as a parameter an angle, that can take any real value. What is interesting is that this space is degenerate since two points 2π apart in the parameter space correspond to the exact same transformation. This means that all points in the $\mathbb{R}^3$ space are *identified* with a set of points contained within a sphere of radius π, so the group manifold of $SO(3)$ is actually a sphere with radius π.

We can choose this sphere to be centered on the origin, so that the range of each of the parameters is given by $\theta^2 + \phi^2 + \psi^2 \leq \pi^2$. At this point we have *almost* defined the group manifold of $SO(3)$. To complete the description we have to note that a rotation by π and $-\pi$ around some axis is equivalent, so the points on the boundary of the ball are identified with their antipodal points.

In conclusion, the group manifold of $SO(3)$ is a ball, of radius π in the parameter space of the matrix representation. The antipodal points of the surface of the ball are connected by an identity, that is, we identify them as the same point. An important consequence of this is that $SO(3)$ is not *simply connected*.

On a simply connected manifold any closed curve can be shrunk to a point by a continuous transformation. If we connect two antipodal points of the ball with some curve, that curve is closed, but cannot be continuously shrunk to a point since its' end points have to remain antipodal or the closed curve will open up. It turns out that if we let a curve run through two adjacent spheres in the parameter space this curve will in fact be shrinkable to a point. To show this we will provide a simple illustration, restricting us to the $\theta - \phi$ axis cross-section of the sphere for graphical simplicity.

Beginning with a straight line from $\phi = -3\pi$ to π, we can continuously deform the line, keeping the points $\phi = -3\pi, -\pi, \pi$ stationary. We can do this until the lines are on the boundary of the sphere as in stage 2 in the image. We then use the identity to transfer the line at the boundary of the extra sphere to the primary sphere. We now have a curve that can be trivially closed by continuous deformation, as is illustrated below



Comparing the manifolds of $SO(3)$ and $SU(2)$ we see that they are very similar, and they also have the same Lie algebra. This connection is no accident, if two groups have sufficiently similar manifolds their Lie algebra is the same. This also shows us a clear example of a group property not given by the Lie algebra, namely *simple connectedness*.

With this introduction to group manifolds we are ready for another definition:

Definition E.3.1. The *direct product*, $G \times H$ between the groups G, H with operators $*, \cdot$ is defined as follows

> Elements in the new group are defined according to the Cartesian product, as $(g, h) : g \in G, h \in H$ on these elements we define the group operator $(\star)$ according to
>
> $$(g,h) \star (g',h') = (g * g',h \cdot h') \ .$$

Some obvious consequences of this definition is that the direct product always has the two constituent groups as subgroups, corresponding to elements of the type g,1 and $(1,h)$. In addition, it is clear that the Lie algebra of the composite group is $\mathbf{g} \oplus \mathbf{h}$ because the subgroups approach the identity independent of each other. The group manifold of the composite group is the Cartesian product of the constituent group manifolds. In the examples in fig. E.1 the isomorphism symbol is used instead of equality because we are only interested in the isomorphism class of a manifold.

Furthermore, the definition of the direct product makes it straightforward to construct a matrix representation of the composite group. Picking two matrix representations of G and H according to T_G, T_H it is easy to see that the block diagonal matrix is in accordance with the definition:

$$\begin{bmatrix} T_G & \mathbf{0} \\ \mathbf{0} & T_H \end{bmatrix} \ .$$

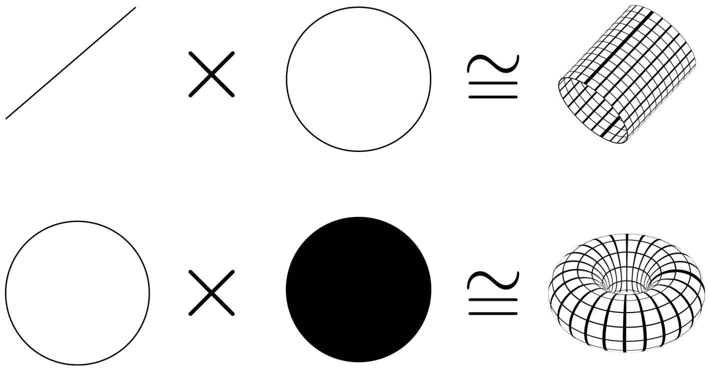


Figure E.1: Two different direct products between manifolds. The direct product of a line and a circle is shown to be a cylinder. The direct product of a circle and a disk embedded in two separate spaces should normally be embedded in 4-space, but it is isomorphic to the solid torus in three dimensions.

E.4 Spin representation

In this thesis we discuss Yang-Mills theory, which is formulated in terms of a group transformation acting on an abstract vector field. In section 2.2 the field was a scalar, and in 3.1 it was unspecified. Apart from the scalar field, classifications include tensor- and spinor fields. In this section we provide a basic discussion on classifying fundamental fields by their *spin j*. A similar classification can also be applied to the symmetry group itself, independent of whether it acts on any fundamental field. In this case, the group is a spin j representation of the special orthogonal group $SO(n{,}k)$. For the gauge group itself to be classified in this manner it must be decomposable into subgroups that are isomorphic to $SO(n{,}k)$.

The importance of this classification lies in quantum physics, where the *spin-statistics theorem* restricts the statistics of a quantized field theory. In particular, quantized spinor fields must obey Fermi-Dirac statistics and quantized scalar and tensor fields must obey Bose-Einstein statistics. That is, a quantized spinor field obeys the Pauli exclusion principle while scalar and tensor fields do not.

The definitions of scalars, spinors and tensors all specify how they transform under a rotation. For some arbitrary field, we can find out how it transforms under a rotation by relating the generators of its gauge group to the generators of the rotation group. For any gauge group, if we can rescale its generators in a way such that it has the same commutation relations as the special orthogonal group (E.5.1), it means that the *group manifolds* of the gauge groups are locally isomorphic. A local isomorphism of the group manifolds is a local equivalence between the transforms that they perform. If this isomorphism exists, we can characterize a fundamental field by performing a rotation infinitesimally.

If we denote this angle of rotation as $\mathbf{\Omega}$, we can define the *spin j* of the fundamental field as the largest j such that

$$\exp\left(\frac{2\pi}{j}T\right) = \mathbb{I} \tag{E.4}$$

where T is the generators of the gauge group in question, rescaled to fulfill the commutation relations of $SO(k{,}n)$.

As an example of this, we use the isomorphism between the gauge groups $SU(2)$ and $SO(3)$. The rotation matrix is given, denoting its generators as T_n and the angle of rotation as Ω, by

$$U_{\mathbf{\Omega}} = \exp\left(\Omega_x T_1 + \Omega_y T_2 + \Omega_z T_3\right).$$

The Pauli matrices E.5.3, with slight modifications fulfill the same commutation relations, so locally a rotation can be expressed as

$$\begin{aligned} U_{\mathbf{\Omega}} &= \exp\left(\frac{i\Omega_x\sigma_1}{2} + \frac{i\Omega_y\sigma_2}{2} + \frac{i\Omega_z\sigma_3}{2}\right) = \exp\left(\frac{i}{2}\begin{bmatrix} \Omega_z & \Omega_x - i\Omega_y \\ \Omega_x + \Omega_y & -\Omega_z \end{bmatrix}\right) \\ &= \begin{bmatrix} \cos\left(\frac{|\Omega|}{2}\right) + i\Omega_z\frac{1}{|\Omega|}\sin\left(\frac{|\Omega|}{2}\right) & i(\Omega_x - i\Omega_y)\frac{1}{\Omega}\sin\left(\frac{|\Omega|}{2}\right) \\ i(\Omega_x + i\Omega_y)\frac{1}{\Omega}\sin\left(\frac{|\Omega|}{2}\right) & \cos\left(\frac{|\Omega|}{2}\right) - i\Omega_z\frac{1}{|\Omega|}\sin\left(\frac{|\Omega|}{2}\right) \end{bmatrix}. \end{aligned}$$

We see that under a rotation Ω by a total of 2π radians the result is

$$U_{\mathbf{\Omega}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and to return to the identity, a rotation by 4π radians is required. From (E.4) we see that the fundamental field coupled to the gauge group $SU(2)$ has spin 1/2. A fundamental field with a half-integer spin is called *spinor*, and $SU(2)$ is referred to as a *spinor representation* of $SO(3)$. In general, this method cannot be applied to a completely general gauge group because isomorphies like $SU(2) \cong SO(3)$ do not nessecarily exist. In a similar fashion it can be shown that $sl(2) \cong so(1{,}2)$ is a spinor representation of $SO(1{,}2)$. For the rotation group of arbitrary order and signature, $SO(k{,}n)$, it's spinor representation is called $Spin(k{,}n)$. In section 6 and onwards we investigate theories that we refer to as spin-2 and spin-3 gravity, respectively. The spin in this refers to the properties of the metric tensor $g_{\mu\nu}$ and the metric-like higher-spin field $\psi_{(\mu\nu\rho)}$. The metric tensor is a symmetric tensor of rank two. A rotation is given locally by $\exp(\Omega T)$. Acting infinitesimally on $g_{\mu\nu}$, we have that

$$\begin{aligned} \delta g_{\alpha\beta} &= \delta\Omega(T^{\mu}_{\alpha}\delta^{\nu}_{\beta} + T^{\nu}_{\beta}\delta^{\mu}_{\alpha})g_{\mu\nu} \\ &= 2\delta\Omega T^{\mu}_{\alpha} g_{\mu\beta} \end{aligned}$$

where symmetry was used to exchange α,β and μ,ν in the second term in the parenthesis. Thus, the rotation generators acting on $g_{\mu\nu}$ act as $\exp(2\Omega T)$, which just a rotation by an angle of 2Ω. We see that the spin of the metric tensor must be two. In the same way we have for $\psi_{(\mu\nu\rho)}$ that

$$\begin{aligned} \delta\psi_{\alpha\beta\sigma} &= \delta\Omega(T^{\mu}_{\alpha}\delta^{\nu\rho}_{\beta\sigma} + (T^{\nu}_{\beta}\delta^{\rho\mu}_{\sigma\alpha} + (T^{\rho}_{\sigma}\delta^{\mu\nu}_{\alpha\beta})\psi_{\mu\nu\rho} \\ &= 3\delta\Omega T^{\mu}_{\alpha}\psi_{\mu\beta\sigma}, \end{aligned}$$

and we see that $\psi_{(\mu\nu\rho)}$ is a spin-3 field.

E.5 Special Lie groups

In this section we will present some of the Lie groups that will be important throughout our work. We will present their generators, Lie algebra and dimensions. We also note some important Lie algebra isomorphisms that are used throughout the thesis.

In general we will refer to groups by their abbreviation, followed by brackets containing the dimension of the group. For example, $SO(5)$ would be the special orthogonal group in five dimensions. If we have two indices inside the brackets as in $SO(m{,}n)$ this means that the group describes a geometry with m time-like dimensions, and n space-like dimensions. The difference between the timelike and space-like dimensions are what metric they have, which will make the representation of $SO(1{,}2)$ different from $SO(3)$ despite them having the same dimension. The generators for $SO(1{,}2)$ are calculated in E.5.1. When referring to the Lie algebra of a Lie group we denote the Lie algebra with the abbreviation of the group in *lowercase* letters. This is standard convention used in order to avoid confusion when referring to Lie groups and Lie algebras simultaneously.

E.5.1 Special orthogonal group, $SO(n)$

The $SO(n)$ group is the special orthogonal group in n dimensions. *Special* denotes a determinant of 1, and *orthogonal* refers to matrix orthogonality.

$SO(2)$

One of the most familiar groups in $SO(n)$ is $SO(2)$, which is the group of rotations in $\mathbb{R}^2$, represented by the single operation

$$U(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} . \tag{E.5}$$

To obtain the Lie algebra we apply equation (E.1) to get

$$T = -[\begin{bmatrix} -\sin(\phi) & -\cos(\phi) \\ \cos(\phi) & -\sin(\phi) \end{bmatrix}_{\phi=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Since $SO(2)$ only contains one generator, it has a trivial Lie algebra.

$SO(3)$

To apply the same method to $SO(3)$ we first need its matrix representation. We can split $SO(3)$ up into a rotation in each orthogonal 2d plane in 3d space, that is, a rotation in the xy-plane, a rotation in the yz-plane and a rotation in the xz-plane.

$$U_0(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix}, U_1(\phi) = \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix}, U_2(\phi) = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{E.6}$$

Which yield the following generators (once again by applying (E.1)):

$$T_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, T_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{E.7}$$

Finally, we find the structure factor by commuting all possible combinations of T_0, T_1, T_2. We will only show two commutators here, and then write down a general expression for the structure factor.

$$[T_0, T_1] = T_0T_1 - T_1T_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T_2$$
$$[T_0, T_2] = T_0T_2 - T_2T_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = -T_1 .$$

These two commutators show the beginnings of a symmetry for the structure factor, and the general formula becomes

$$[T_i, T_j] = \epsilon_{ijk}T_k . \tag{E.8}$$

We see that the structure constants $f_{ijk} = \epsilon_{ijk}$ as per the notation in eq (E.3).

$SO(1,2)$

$SO(1,2)$ is the special orthogonal group with associated metric

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Inutitively, we want to have one length-preserving parametric transformation for every orthogonal plane we can find in $\mathbb{R}^n$. If a plane consists of one space-like coordinate and one time-like coordinate then the length preserving transformation is the Lorentz transformation instead of the rotation that we have seen before.

The regular Lorentz transformation with an inertial system S and a boosted system S', S' travelling with velocity v with respect to S, is defined as follows (in x variable and t variable):

$$x' = \gamma(x - vt) \ , \tag{E.9}$$

$$t' = \gamma(t - vx/c^2) \ , \tag{E.10}$$

where γ is the Lorentz factor, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

If we introduce the hyperbolic parameter θ, also called *rapidity*, defined as $\mathrm{e}^\theta = \gamma(1 + v/c)$ we may rewrite the Lorentz transform [12]. We compute $\sinh\theta$ and $\cosh\theta$ for reasons which will become apparent soon. We do moreover, from now and on, set $c = 1$ for simplicity:

$$\sinh\theta = \frac{\mathrm{e}^\theta - \mathrm{e}^{-\theta}}{2} = \frac{\gamma(1+v) - 1/(\gamma(1+v))}{2} = \frac{\gamma^2(1+v)^2 - 1}{2\gamma(1+v)} = \frac{1+v-(1-v)}{2\gamma(1-v^2)} = v\gamma \ ,$$

and

$$\cosh\theta = \frac{\mathrm{e}^\theta + \mathrm{e}^{-\theta}}{2} = \frac{\gamma(1+v) + 1/(\gamma(1+v))}{2} = \frac{\gamma^2(1+v)^2 + 1}{2\gamma(1+v)} = \frac{1+v+(1-v)}{2\gamma(1-v^2)} = \gamma \ .$$

Now we can express the Lorentz transform on hyperbolic form, yielding:

$$x' = x\cosh(\theta) - t\sinh(\theta) \ , \tag{E.11}$$

$$t' = t\cosh(\theta) - x\sinh(\theta) \ . \tag{E.12}$$

Equation (E.11) is very conveniently dependent on a parameter θ so that we can apply equation (E.1). The transformation matrices for $SO(1{,}2)$ are then simply:

$$U_0(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} , U_1(\theta) = \begin{bmatrix} \cosh(\theta) & -\sinh(\theta) & 0 \\ -\sinh(\theta) & \cosh(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} , U_2(\theta) = \begin{bmatrix} \cosh(\theta) & 0 & -\sinh(\theta) \\ 0 & 1 & 0 \\ -\sinh(\theta) & 0 & \cosh(\theta) \end{bmatrix} . \tag{E.13}$$

Inserting into equation (E.1) we get the generators

$$T_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} , T_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , T_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} . \tag{E.14}$$

We find the structure factor to be ${f_{ij}}^k = -\epsilon_{ij}^k$. To lower the k we need to note that for $k = 0$ we get an extra minus sign because the manifold of the group has a Minkowski metric. This we see is the only difference between the Lie algebra of $SO(3)$ and $SO(1{,}2)$.

E.5.2 Special linear group, $SL(n)$

$SL(n)$ is the group of $n \times n$ matrices with determinant 1. We will explicitly calculate the Lie algebra for $n = 2{,}3$. To find the Lie algebrae of $SL(2)$ and $SL(3)$ we will use LU-factorization, meaning we split the the arbitrary square matrix into a lower triangular matrix and an upper triangular matrix. This operation in reality depends on none of the diagonal elements of the matrix in question being 0, so it is only valid around the identity. Also, the decomposition actually adds n degrees of freedom, so we must add another restriction to obtain a unique decomposition. A valid such restriction is the requirement that the lower triangular matrix must be a unit triangular matrix, that is all of its diagonal elements are 1.

$SL(2)$

We write down $SL(2)$ as the product of a lower- and upper triangular matrix right away:

$$g = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} r & b \\ 0 & r^{-1} \end{bmatrix} ,$$

where the rs come from the determinant condition. As usual, using (E.1) we obtain the generators:

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad S_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad .$$

To obtain a simple form of the commutation relations we rescale the generators as

$$S_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad S_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad S_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad .$$

making the commutation relations

$$[S_0,S_1] = S_1 \quad , \quad [S_2,S_0] = S_2 \quad , \quad [S_1,S_2] = S_0 \quad .$$

We can recombine the generators of $SL(2)$ so that its isomorphism to $SO(1,2)$ is readily apparent. We define a new set of generators as:

$$\begin{aligned} T_0 = S_0 &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \\ T_1 = \frac{S_1 + S_2}{\sqrt{2}} &= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \\ T_2 = \frac{S_1 - S_2}{\sqrt{2}} &= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \end{aligned}$$

We evaluate the commutators of rising order and then write down the general expression:

$$\begin{aligned} [T_0,T_1] &= \frac{1}{\sqrt{2}}[S_0,S_1 + S_2] \\ &= \frac{S_1 - S_2}{\sqrt{2}} = T_2 , \\ [T_2,T_0] &= \frac{1}{\sqrt{2}}[S_1 - S_2,S_0] \\ &= -\frac{S_2 + S_3}{\sqrt{2}} = -T_1 , \\ [T_1,T_2] &= \frac{1}{2}[S_1 + S_2,S_1 - S_2] , \\ &= \frac{1}{2}\left([S_2,S_1] - [S_1,S_2]\right) = \frac{1}{2}(-S_0 - S_0) = -T_0 . \end{aligned}$$

Going off of these commutators we can write down the commutation relations as

$$[T_a,T_b] = \epsilon_{ab}{}^{c}T_c ,$$

where, as usual, the c can be lowered by the Minkowski metric, however in this case the minus sign is attached to the three. For the Minkowski metric to follow the convention of the rest of this appendix we switch the names of T_0 and T_2 so that our final generators are:

$$T_0 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad T_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad T_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

and the structure constants are still $\epsilon_{ab}{}^{c}$.

$SL(3)$

We will find the Lie algebra of $SL(3)$, and then show how to find a diagonal embedding of $SL(2)$ in the larger $SL(3)$ gauge group. For ease of finding, we have split off the other possible recombinations of the $SL(3)$ Lie algebra to their own Appendix, E.5.2. We do not try to show how to find these particular recombinations, but rather just present their matrices and commutation relations. These recombinations specify a *principal embedding*, where the commutation relations between the $sl(2)$ subalgebra and the five remaining generators are non-trivial.

We write down the LU-decomposition of a three-dimensional matrix with a determinant equal to one:

$$g = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} , \begin{bmatrix} k^{-1} & d & e \\ 0 & kr & f \\ 0 & 0 & r^{-1} \end{bmatrix} .$$

The generators can easily be read off, as the off-diagonal parameters are entirely free:

$$T_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad T_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

$$T_a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad T_b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} ,$$

$$T_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \quad T_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$T_e = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad T_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

where we number the matrices by the usual reading order, and the subscript indicates from which parameter the generator was obtained.

Performing a recombination according to:

$$T_0 = -\frac{1}{2}T_k = \frac{1}{2}\begin{bmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$T_1 = \frac{T_d + T_a}{2} = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$T_2 = \frac{T_d - T_a}{2} = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

we see that these three generators T_1, T_2, T_3 obviously satisfy the commutation relations in E.5.2. Finding the matrices fulfilling the $sl(2)$ commutation relations by replicating the $sl(2)$ algebra in 2 our of the three indices of $sl(3)$ is called the *diagonal* embedding. The remaining five generators $W_{1\ldots5}$ are just $T_k - T_r$ together with T_b, T_c, T_e, T_f. The most important property of this embedding is that the higher spin generators W_a have trivial commutation relations with the $SL(2)$ generators T_b. Listing a preliminary set of higher spin generators in a diagonal embedding, we have

$$W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} , \quad W_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \quad W_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ,$$

$$W_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad W_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

Since we do not use this embedding in our thesis, we do not present any commutation relations. It is likely that the higher spin generators W should be recombined to obtain simple commutation relations before they are employed in an actual gauge theory. The purpose of showing these generators is to illustrate the difference between a diagonal and a principal embedding, such as the recombinations of $SL(3)$ that we will state now.

Conventions for Lie algebras in spin-3

Here we state the explicit matrix form of the different generators used in higher spin calculations. We follow the conventions of [32]. We also give the Lie algebra and commutators. The standard $sl(3)$ algebra is generated by T_a and T_{ab} where T_a forms a $sl(2)$ subalgebra.

$$\begin{aligned}
[T_a,T_b] &= \epsilon_{ab}{}^{c}T_c \ , \\
[T_a,T_{bc}] &= 2\epsilon^{d}{}_{a(b}T_{c)d} \ , \\
[T_{ab},T_{cd}] &= -2\left(\eta_{a(c}\epsilon_{d)b}{}^{e} + \eta_{b(c}\epsilon_{d)a}{}^{e}\right)T_e \ .
\end{aligned}$$

The invariant bilinear form denoted by "tr" gives

$$\begin{aligned}
\mathrm{tr}[T_aT_{ab}] &= 2\eta_{ab} \ , \\
\mathrm{tr}[T_aT_{bc}] &= 0 \ , \\
\mathrm{tr}[T_{ab}T_{cd}] &= -\frac{4}{3}\eta_{ab}\eta_{cd} + 2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \ .
\end{aligned}$$

It is useful to define new generators, L_i and W_j, by linearly combining T_a and T_{ab} according to

$$\begin{aligned}
T_0 &= \frac{1}{2}(L_1 + L_{-1}) \ , & T_1 &= \frac{1}{2}(L_1 - L_{-1}) \ , & T_2 &= L_0 \ , \\
T_{00} &= \frac{1}{4}(W_2 + W_{-2} + 2W_0) \ , & T_{01} &= \frac{1}{4}(W_2 - W_{-2}) \ , & T_{02} &= \frac{1}{2}(W_1 + W_{-1}) \ , \\
T_{11} &= \frac{1}{4}(W_2 - W_{-2} - 2W_0) \ , & T_{12} &= \frac{1}{2}(W_1 - W_{-1}) \ , & T_{22} &= W_0 \ .
\end{aligned}$$

The generators L and W obey the following commutation relations

$$\begin{aligned}
[L_i,L_j] &= (i-j)L_{i+j} \ , \\
[L_i,W_j] &= (2i-j)Wi+j \ , \\
[W_i,W_j] &= -\frac{1}{3}(i-j)(2i^2+2j^2-ij-8)L_{i+j} \ .
\end{aligned}$$

and all the non-zero components of the invariant bilinear form are

$$\begin{aligned}
\mathrm{tr}[L_0L_0] &= 2, & \mathrm{tr}[L_1L_{-1}] &= -4 \ , & & \\
\mathrm{tr}[W_0W_0] &= \frac{8}{3}, & \mathrm{tr}[W_1W_{-1}] &= -4 \ , & \mathrm{tr}[W_2W_{-2}] &= 16 \ .
\end{aligned}$$

The explicit matrix representation of the generators L_i and W_j is

$$\begin{aligned}
L_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ , & L_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \ , & L_{-1} &= \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \ , \\
W_0 &= \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ , & W_1 &= \frac{2}{3}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \ , & W_2 &= 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ , \\
W_{-1} &= \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \ , & W_{-2} &= 2\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ .
\end{aligned}$$

E.5.3 Special unitary group, $SU(n)$

The $SU(n)$ group is the group of unitary matrices with determinant 1. To find the generators for some arbitrary matrix A that fulfills $A^\dagger A = 1$ we will begin by observing that such a matrix can be written as its *Cholesky decomposition*:

$$A = U^\dagger U \ , \tag{E.15}$$

where U is an upper diagonal matrix. This makes enforcing the determinant equal to one condition a lot easier then for an arbitrary general $n \times n$ matrix A. The diagonal of U will have $n-1$ degrees of freedom since we have an equation of n real variables equal to a constant. Off the diagonal we have two degrees of freedom per index, for a total of $2 \cdot \frac{1}{2}((n-1)^2 + n - 1)$. The sum of the degrees of freedom is then $n^2 - 1$.

SU(2)

$SU(2)$ is a convenient example to start with due to its simplicity and importance in particle physics. The symmetries of the group play a crucial role when describing electroweak interaction. $SU(2)$ is represented by matrices of rank 2 and determinant 1. These matrices are also unitary. From the fact that $SU(2)$ is *special*, i.e. its group elements have determinant 1, it follows that the trace of the generators of the group are zero. This can be shown by using the exponential relationship between the generators (T_i), free parameters θ^i and the group elements U:

$$U(\theta^i) = \mathrm{e}^{\theta^i T_i} \ , \tag{E.16}$$

and the following identity[1]

$$\det(\mathrm{e}^R) = \mathrm{e}^{\mathrm{tr}\,[R]} \ , \tag{E.17}$$

where R is a matrix.

If we substitute $R = \theta^i T_i$ we find

$$\det(\mathrm{e}^{\theta^i T_i}) = \det(U) = 1 = \mathrm{e}^{\mathrm{tr}[\theta^i T_i]}. \Rightarrow \mathrm{tr}[T_i] = 0 \ , \tag{E.18}$$

where we used the linearity property of the trace. We moreover note that the generators T_i are *anti-hermitian*. This follows directly from expanding the exponential in equation (E.16) to first order and compute the product $U^\dagger U$:

$$U^\dagger U \simeq (1 + \theta^i T_i^\dagger)(1 + \theta^i T_i) = 1 + \theta^i (T_i + T_i^\dagger) + ... = 1 \Rightarrow T_i = -T_i^\dagger \ . \tag{E.19}$$

Thus we conclude that these generators of $SU(2)$ are both traceless *and* anti-hermitian.[2] In $SU(2)$ the generators are therefore given of the form

$$T_i = \begin{bmatrix} \alpha & \beta \\ \beta^* & -\alpha \end{bmatrix} \ , \tag{E.20}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. By setting $\alpha = 0$ we find two different kind of generators:

$$T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \tag{E.21}$$

and

$$T_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ . \tag{E.22}$$

The third and last generator of $SU(2)$ we get by setting $\beta = 0$:

$$T_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ . \tag{E.23}$$

[1] This identity can be shown by triangulizing R, i.e let $R = PTP^{-1}$, where T is upper-triangular with eigenvalues $\lambda_1 ... \lambda_n$ on the diagonal. Then it follows that e^T is a diagonal matrix with $\mathrm{e}^{\lambda_1} ... \mathrm{e}^{\lambda_n}$ on the diagonal. Since the determinant is the product of eigenvalues it directly follows that $\det(\mathrm{e}^T) = \mathrm{e}^{\mathrm{tr}\,T}$. Finally we observe that R and T have the same eigenvalues, thus we must have $\mathrm{tr}\,R = \mathrm{tr}\,T$. Moreover $P\mathrm{e}^T P^{-1} = \mathrm{e}^R$ (from $PT^kP^{-1} = R^k$ for all k) and therefore $\det(\mathrm{e}^R) = \det(\mathrm{e}^T) = \mathrm{e}^{\mathrm{tr}\,T} = \mathrm{e}^{\mathrm{tr}\,R}$ and we have proven our identity.

[2] Particle physicists often tend to choose a different representation of $SU(2)$, where the matrices are hermitian.

Note that the derived generators are not unique, however they form the smallest non-trivial representation of the Lie algebra, also called the *fundamental* representation which is unique for $SU(2)$. The generators T_1, T_2 and T_3 are called *Pauli matrices* and are commonly denoted σ_1, σ_2 and σ_3. From now on we stick to conventional notation and simply let $T_1 = \sigma_1$, $T_2 = \sigma_2$ and $T_3 = \sigma_3$. To derive the Lie algebra to $SU(2)$ we simply compute the commutators $[\sigma_1,\sigma_2], [\sigma_2,\sigma_3]$ and $[\sigma_1,\sigma_3]$ (the rest follow in the same manner):

$$[\sigma_1,\sigma_2] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2i\sigma_3 \ ,$$

$$[\sigma_2,\sigma_3] = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i\sigma_1 \ ,$$

$$[\sigma_1,\sigma_3] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = 2i\sigma_2 \ .$$

We can thus summarize the Lie algebra of $SU(2)$ according to[3]

$$[T_a,T_b] = [\sigma_a,\sigma_b] = 2i\epsilon_{abc}\sigma_c \ , \tag{E.24}$$

where ϵ_{abc} is the completely antisymmetric Levi-Civita symbol and the structure constant(s). From the Lie algebra of $SO(3)$ and the commutator relation in equation (E.8) we deduce that the structure factors of $SO(3)$ and $SU(2)$ only differ in sign why we may conclude that $SU(2)$ is isomorphic to $SO(3)$.

$SU(3)$

We may generalize the discussion in the last section by considering the group $SU(3)$. The matrix U can be written as:

$$U = \begin{bmatrix} a & d+i\rho & e+i\epsilon \\ 0 & b & f+it \\ 0 & 0 & c \end{bmatrix} \ . \tag{E.25}$$

We then use the restriction $abc = 1$ to rewrite (E.25) as $a = cr, b = r^{-1}, c = c^{-2}$. To get a matrix to plug into equation (E.1) we just need to evaluate $U^\dagger U$. We will do this and evaluate the partial derivatives in each of the parameters around the unit matrix ($r = c = 1, e = \epsilon = d = \rho = f = t = 0$).

$$\begin{aligned} U^\dagger U = A &= \begin{bmatrix} cr & 0 & 0 \\ d-i\rho & r^{-1} & 0 \\ e-i\epsilon & f-it & c^{-1} \end{bmatrix} \begin{bmatrix} cr & d+i\rho & e+i\epsilon \\ 0 & r^{-1} & f+it \\ 0 & 0 & c^{-1} \end{bmatrix} \\ &= \begin{bmatrix} c^2r^2 & cr(d+i\rho) & cr(e+i\epsilon) \\ (d-i\rho)cr & d^2+\rho^2+r^{-2} & (d-i\rho)(e+i\epsilon)+r^{-1}(f+it) \\ (e-i\epsilon)cr & (e-i\epsilon)(d+i\rho)+(f-it)r^{-1} & e^2+\epsilon^2+f^2+t^2+c^{-2} \end{bmatrix} \ . \end{aligned}$$

We continue by evaluating the partial derivatives around the identity matrices, defining $\mathbf{V} = \{r,c,e,\epsilon,d,\rho,f,t\}$ with $A(V_0)$ defining the identity matrix.

$$T_1 = -i\left[\frac{\partial A}{\partial r}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \qquad T_2 = -i\left[\frac{\partial A}{\partial c}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \ ,$$

$$T_3 = -i\left[\frac{\partial A}{\partial d}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \qquad T_4 = -i\left[\frac{\partial A}{\partial \rho}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ ,$$

$$T_5 = -i\left[\frac{\partial A}{\partial e}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ , \qquad T_6 = -i\left[\frac{\partial A}{\partial \epsilon}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \ ,$$

$$T_7 = -i\left[\frac{\partial A}{\partial f}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \ , \qquad T_8 = -i\left[\frac{\partial A}{\partial t}\right]_{\mathbf{V}=V_0} = -i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix} \ .$$

[3] One can get rid of the factor 2 in the commutator by normalizing the generators T_i such that $T_i = \frac{\sigma_i}{2}$. This is conventionally done to yield a neater expression.

It is very common to rescale this Lie algebra to another form so that the algebra is more similar to that of $SU(2)$. Replacing S_1, S_2 with $T_1 = \frac{1}{2}S_1$ and $T_2 = \frac{2S_2 - S_1}{2\sqrt{3}}$ one gets the *Gell-Mann matrices.* This particular set of generators (which clearly spans all of $SU(3)$) were chosen by the American physicist Murray Gell-Mann (1929-) because they naturally extend the Pauli matrices from $SU(2)$ to $SU(3)$, which formed the basis for his model of quarks [45].

$SO(2,2)$

The Lie group $SO(2,2)$ consisting of all 4×4 matrices with determinant 1 and that forfill the orthogonality relation $X^\dagger \eta X = \eta$ with respect to the Lorentzian metric $\eta = \mathrm{diag}(-1,1,1,1)$ is of great importance in fundamental physics as it is the isometry group of AdS_3, the three-dimensional Anti de-Sitter space. In this thesis we make use of the fact that the Lie algebra of $SO(2,2)$ is isomorphic to that of $SO(1,2) \times SO(1,2)$ when relating the Chern-Simons gauge action to Einstein-Hilbert action. From the orthogonality relation for an element X in $SO(2,2)$ we may derive a condition for the generators of $SO(2,2)$ by expanding the exponential definition of the group element:

$$X = \exp(\theta^i A_i) \Rightarrow X \simeq 1 + \theta^i A_i \ . \tag{E.26}$$

For simplicity we consider only real elements X and generators A_i. Plugging the expansion into the orthogonality relation we find

$$X^T \eta X = (1 + \theta^i A_i^T)\eta(1 + \theta^i A_i) = \eta \Rightarrow A_i^T \eta + \eta A_i = 0 \ . \tag{E.27}$$

Moreover, since $SO(2,2)$ is a special group its generators are traceless. We have therefore two conditions which have to be satisfied by the generators. We do now perform a block decomposition of our generators A_i according to

$$A_i = \begin{bmatrix} S & T \\ U & V \end{bmatrix} \ ,$$

where S,T,U and V are real 2×2 matrices. Substituting the decomposition into our derived equation (I.3) we find

$$\begin{bmatrix} S^T & U^T \\ T^T & V^T \end{bmatrix} \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} + \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} S & T \\ U & V \end{bmatrix} = \begin{bmatrix} -S^T - S & U^T - T \\ -T^T + U & V^T + V \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ .$$

From this relation we may rewrite our generators as follows

$$A_i = \begin{bmatrix} S & T \\ T^T & V \end{bmatrix} \ ,$$

and we find that S,T, U and V must obey the following relations

$$\begin{cases} U = T^T \\ V = -V^T \\ S = -S^T \end{cases} \ .$$

From these conditions combined with the condition $\mathrm{tr}\,[A_i] = 0$ it is possible to find six different generators in a fundamental representation. This should be no surprise since there are six orthogonal planes in 4 dimensions to Lorentz boost and rotate in. We list these generators:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \ .$$

It is a quite tedious task to compute the commutators $[A_i,A_j]$ in order to derive the Lie algebra of $SO(2,2)$ so this is done numerically. However, at this point we are settled for showing the isomorphism between $SO(2,2)$ and $SO(1,2) \times SO(1,2)$, since this result is of importance in our thesis.

What does it even mean that the algebra of $SO(2{,}2)$ is isomorphic to that of $SO(1{,}2) \times SO(1{,}2)$? In proper English we have verified the statement if we can show that $SO(2{,}2)$ can be separated into two parts, each part having its own subalgebra, in this case that of $SO(1{,}2)$. To prove this it will be useful to consider linear combinations of the derived generators A_i, $i = 1...6$. Consider the cleverly chosen linear combinations[4]

$$\begin{cases} A_{11} = \frac{1}{2}(A_1 + A_2),\ A_{21} = \frac{1}{2}(A_1 - A_2) \\ A_{12} = \frac{1}{2}(X + Y),\ A_{22} = \frac{1}{2}(X - Y) \\ A_{13} = \frac{1}{2}(P + Q),\ A_{23} = \frac{1}{2}(P - Q) \end{cases} ,$$

where X, Y, P and Q are yet unknown 4×4 matrices and the factor $\frac{1}{2}$ a normalization factor. We begin with the condition $[A_{1i}, A_{2j}] = 0$ and construct our unknowns A_{12}, A_{22}, A_{13} and A_{23} from it. Starting off by evaluating $[A_{11}, A_{22}]$ we obtain:

$$[A_{11}, A_{22}] = \frac{1}{4}([A_1, X] + [A_1, -Y] + [A_2, X] + [A_2, -Y]) \ .$$

We now make the "arbitrary" choice $X = A_3$ leading to

$$[A_{11}, A_{22}] = \frac{1}{4}([A_1, A_3] + [A_1, -Y] + [A_2, A_3] + [A_2, -Y]) = \frac{1}{4}(-A_4 + [A_1, -Y] - A_5 + [A_2, -Y]) \ ,$$

where we used $[A_1, A_3] = -A_4$ and $[A_2, A_3] = -A_5$. In order to achieve $[A_{11}, A_{22}] = 0$ we must now choose $Y = -A_6$, due to the fact that $[A_1, A_6] = +A_5$ and $[A_2, A_6] = +A_4$. Thus we have $X = A_3$ and $Y = -A_6$ and deduce $A_{12} = \frac{1}{2}(A_3 - A_6)$ and $A_{22} = \frac{1}{2}(A_3 + A_6)$. We use the same procedure to solve for P and Q. Consider

$$[A_{13}, A_{22}] = \frac{1}{4}([P, A_3] + [P, A_6] + [Q, A_3] + [Q, A_6]) \ ,$$

and let $P = A_4$, leading to

$$\frac{1}{4}([A_4, A_3] + [A_4, A_6] + [Q, A_3] + [Q, A_6]) = \frac{1}{4}(-A_1 + A_2 + [Q, A_3] + [Q, A_6]) \ ,$$

and to get $[A_{13}, A_{22}] = 0$ we therefore have to choose $Q = A_5$:

$$\frac{1}{4}(-A_1 + A_2 + [A_5, A_3] + [A_5, A_6]) = \frac{1}{4}(-A_1 + A_2 + A_1 - A_2) = 0 \ .$$

Thus we have found $A_{13} = \frac{1}{2}(A_4 + A_5)$ and $A_{23} = \frac{1}{2}(A_4 - A_5)$. Well, is all work done now? No, we obviously need to check that all commutators $[A_{1i}, A_{2j}]$ vanish and not just a few of them. This is done numerically and amazingly the relation holds for our seemingly non-arbitrary choices of $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}$ and A_{23}! Thus we have shown that A_{1i} and A_{2j} can be separated and form their own subalgebra. By forming the commutators $[A_{1i}, A_{1j}]$ and $[A_{2i}, A_{2j}]$ we find

$$[A_{1i}, A_{1j}] = -\epsilon_{ij}{}^k A_{1k} \ ,$$
$$[A_{2i}, A_{2j}] = -\epsilon_{ij}{}^k A_{2k} \ ,$$

and this is precisely the Lie algebra of $SO(1{,}2)$ (see section E.5.1). Thus A_{1i} and A_{2j} each constitute a $SO(1{,}2)$ algebra.

[4] the choices of combinations are inspired by Thyssen and Ceulemans *Shattered Symmetry: Group Theory From the Eightfold Way to the Periodic Table*, page 307 [46].