Review: Brownian motion

- A process $\{B_t\}_{t\geq 0}$ where $B_t \sim \text{Normal}(0, t)$. No parameters.
- Independent normally distributed increments.
- Continuous paths that are nowhere differentiable.
- Connection to random walks: The Donsker principle.
- Gaussian processes.
- Restarting Brownian motions at stopping times.

The distribution of the first hitting time

- Given $a \neq 0$ what is the distribution of the first hitting time $T_a = \min\{t : B_t = a\}$?
- We prove below that

$$\frac{1}{T_a} \sim \text{Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$$

- Assuming that $a > 0$ and using that T_a is a stopping time we get for any $t > 0$ that $\Pr\left(B_{1/t} > a \mid T_a < 1/t\right) = \Pr\left(B_{1/t - T_a} > 0\right) = \frac{1}{2}$.
- We also have

$$\Pr\left(B_{1/t} > a \mid T_a < 1/t\right) = \frac{\Pr\left(B_{1/t} > a, T_a < 1/t\right)}{\Pr\left(T_a < 1/t\right)} = \frac{\Pr\left(B_{1/t} > a\right)}{\Pr\left(T_a < 1/t\right)}.$$

- It follows that $\Pr\left(T_a < 1/t\right) = 2\Pr\left(B_{1/t} > a\right)$ and so

$$\Pr\left(\frac{1}{T_a} < t\right) = 2\Pr\left(B_{1/t} < a\right) - 1 = 2\Pr\left(B_1 < at^{1/2}\right) - 1.$$

- Taking the derivative w.r.t. t we get the Gamma density

$$\pi_{1/T_a}(t) = 2\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(at^{1/2})^2\right) \frac{a}{2} t^{-1/2}.$$

Maximum of Brownian motion

- Define $M_t = \max_{0 \leq s \leq t} B_s$.
- We may compute for $a > 0$ (using result from previous page)

$$\Pr(M_t > a) = \Pr(T_a < t) = 2\Pr(B_t > a) = \Pr(|B_t| > a)$$

- Thus M_t has the same distribution as $|B_t|$, the absolute value of B_t.
- Example: What is the probability that $M_3 > 5$?
- Example: Find t such that $\Pr(M_t \leq 4) = 0.9$.

MVE550 2022 Lecture 13
Dobrow Sections 8.4 - 8.6
Zeros of Brownian motion. Brownian bridge.
Modelling stocks and options.
Martingales. Black Scholes

Petter Mostad

Chalmers University

December 13, 2022

Zeros of Brownian motion

- Let L be the *last zero* in $(0,1)$ of Brownian motion. (In other words, $L = \max\{t : 0 < t < 1, B_t = 0\}$. Then
$$L \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right).$$
- Outline of proof on next page.
- Consequence: Let L_t be the last zero in $(0,t)$. Then
$$L_t/t \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right).$$
- Note: The probability that Brownian motion has at least one zero in (r,t) for $0 \leq r < t$ is $1 - \Pr(L_t < r)$.
- Note: The cumulative distribution for the Beta density can be computed with the arcsin function:
$$\Pr(L_t < r) = \int_0^{r/t} \text{Beta}\left(s; \frac{1}{2}, \frac{1}{2}\right)\, ds = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{r}{t}}\right)$$

Outline of proof

$$\begin{aligned}
\Pr(L > s) &= \int_{-\infty}^{\infty} \Pr(L > s \mid B_s = t)\, \mathrm{Normal}(t; 0, s)\, dt \\
&= 2\int_{0}^{\infty} \Pr(M_{1-s} > t)\, \mathrm{Normal}(t; 0, s)\, dt \\
&= 2\int_{0}^{\infty} 2\Pr(B_{1-s} > t)\, \mathrm{Normal}(t; 0, s)\, dt \\
&= 4\int_{0}^{\infty}\int_{t}^{\infty} \mathrm{Normal}(r; 0, 1-s)\, \mathrm{Normal}(t; 0, s)\, dr\, dt \\
&= \ldots \\
&= \frac{1}{\pi}\int_{s}^{1} \frac{1}{\sqrt{x(1-x)}}\, dx \\
&= \int_{s}^{1} \mathrm{Beta}\left(x; \frac{1}{2}, \frac{1}{2}\right)\, dx
\end{aligned}$$

Brownian bridge

- Define a Gaussian process X_t by conditioning Brownian motion B_t on $B_1 = 0$. Then X_t is a *Brownian bridge*.
- If $0 < s < t < 1$ then (B_s, B_t, B_1) is multivariate normal with

$$\mathrm{E}\left((B_s, B_t, B_1)\right) = (0, 0, 0), \quad \mathrm{Var}\left((B_s, B_t, B_1)\right) = \Sigma = \begin{bmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{bmatrix}.$$

  Conditioning on $B_1 = 0$ and using properties of the multivariate normal (or see Dobrow) we get $\mathrm{E}\left(X_t\right) = 0$ and

$$\mathrm{Cov}\left(X_s, X_t\right) = s - st.$$

- Define another Gaussian process with $Y_t = B_t - tB_1$. Then we see that $\mathrm{E}\left(Y_t\right) = 0$ and (when $0 < s < t < 1$)

$$\mathrm{Cov}\left(Y_s, Y_t\right) = s - st.$$

  It follows that this is identical to the Brownian bridge defined above.
- Example: Estimate by simulation: If a Brownian motion fulfills $B_1 = 0$, what is the probability that it has values below -1?

Brownian motion with a drift

- For real μ and $\sigma > 0$ define the Gaussian process X_t as

$$X_t = \mu t + \sigma B_t$$

  This is *Brownian motion with a drift*, and is often a more useful model than standard Brownian motion.
- Examples:
  - The amount won or lost in a game of chance that is not fair (approximating discrete winnings / losses with continuous changes).
  - The score difference between two competing sports teams (approximating this difference with a continuous function).
- This is a Gaussian process with continuous paths and stationary and independent increments.
- Example: Computing the chance of winning team game based on intermdiate score.
- Note: If a Brownian motion with drift is observed at points $y_1, \ldots, y_n$ and μ and σ are not fixed, there are priors so that we can do conjugate analysis, and analytically get a posterior process. However this posterior process is not a Gaussian process.

Geometric Brownian motion

- The stochastic process

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  where $G_0 > 0$ is called *geometric Brownian motion* with drift parameter μ and variance parameter σ^2.
- $\log(G_t)$ is a Gaussian process with expectation $\log(G_0) + \mu t$ and variance $t\sigma^2$.
- Show that
  - $\mathrm{E}(G_t) = G_0 e^{t(\mu + \sigma^2/2)}$
  - $\mathrm{Var}(G_t) = G_0^2 e^{2t(\mu+\sigma^2/2)}(e^{t\sigma^2} - 1)$
- Natural model for things that develop by multiplication of random independent factors, rather than addition of random independent increments. Example: Stock prices.

Modelling stock price with geometric Brownian motion

- To model the price of a stock, it is reasonable to
  - use a continuous-time stochastic model.
  - consider the *factor* with which it changes, not the differences in prices.
  - consider normal distributions for such factors (?)
  - use a parameter for the trend of the price, and one for the variability of the price.
  - make a Markov assumption(???)
- This leads to using a geometric Brownian motion as model

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  In this context σ is called the *volatility* of the stock.
- Example: A stock price is modelled with $G_0 = 67.3$, $\mu = 0.08$, $\sigma = 0.3$. What is the probability that the price is above 100 after 3 years?

Discounting future values

- When making investments, there is always a range of choices, some of which are sometimes called “risk free”. Such investments may pay a fixed interest.
- When interests are compounded frequently, a reasonable model is that an investment of G_0 has a value $G_0 e^{rt}$ after time t, where r is the “risk free” investment rate of return.
- A common way to take this alternative into account is to instead “discount” all other investments with the factor e^{-rt}.
- For example, the value of a stock may be modelled with

$$e^{-rt} G_t = e^{-rt} G_0 e^{\mu t + \sigma B_t} = G_0 e^{(\mu - r)t + \sigma B_t}$$

  So discounting corresponds to adjusting the trend parameter from μ to $\mu - r$.

Stock options

- A (European) stock option is a right (but not obligation) to buy a stock at a given time t in the future for a given price K.
- How much can you expect to earn from a stock option at that future time?
- We get that (see next page)

$$\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 e^{t(\mu+\sigma^2/2)} \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

  where $\beta = (\log(K/G_0) - \mu t)/\sigma$.
- Example: A stock price is modelled with $G_0 = 67.3$, $\mu = 0.08$, $\sigma = 0.3$. What is the expected payoff from an option to buy the stock at 100 in 3 years?

Proof

- Prove the algebraic identity

$$e^{\sigma x}\,\mathrm{Normal}(x; 0, t) = e^{\sigma^2 t/2}\,\mathrm{Normal}(x; \sigma t, t)$$

- Then, defining $\beta = (\log(K/G_0) - \mu t)/\sigma$, we get

$$\begin{aligned}
\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) &= \mathrm{E}\left(\max\left(G_0 e^{\mu + \sigma B_t} - K, 0\right)\right) \\
&= \int_{-\infty}^{\infty} \max\left(G_0 e^{\mu t + \sigma x} - K, 0\right) \mathrm{Normal}(x; 0, t)\, dx \\
&= \int_{\beta}^{\infty} \left(G_0 e^{\mu t + \sigma x} - K\right) \mathrm{Normal}(x; 0, t)\, dx \\
&= G_0 e^{\mu t} \int_{\beta}^{\infty} e^{\sigma x}\, \mathrm{Normal}(x; 0, t)\, dx - K \int_{\beta}^{\infty} \mathrm{Normal}(x; 0, t)\, dx \\
&= G_0 e^{t(\mu + \sigma^2/2)} \int_{\beta}^{\infty} \mathrm{Normal}(x; \sigma t, t)\, dx - K \int_{\beta}^{\infty} \mathrm{Normal}(x; 0, t)\, dx \\
&= G_0 e^{t(\mu + \sigma^2/2)} \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)
\end{aligned}$$

Martingales

- A stochastic process $(Y_t)_{t\geq 0}$ is a *martingale* if for $t \geq 0$
  - $\mathrm{E}(Y_t \mid Y_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) < \infty$.
- Brownian motion is a martingale.
- $(Y_t)_{t\geq 0}$ is a *martingale with respect to* $(X_t)_{t\geq 0}$ if for all $t \geq 0$
  - $\mathrm{E}(Y_t \mid X_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) < \infty$.
- Example: Define $Y_t = B_t^2 - t$ for $t \geq 0$. Then Y_t is a martingale with respect to Brownian motion.

Geometric Brownian motion can be a martingale

Let G_t be Geometric Brownian motion. We get

$$\begin{aligned}
& \mathrm{E}\left(G_t \mid B_r, 0 \le r \le s\right) \\
= \;& \mathrm{E}\left(G_0 e^{\mu t + \sigma B_t} \mid B_r, 0 \le r \le s\right) \\
= \;& \mathrm{E}\left(G_0 e^{\mu(t-s) + \sigma(B_t - B_s)} e^{\mu s + \sigma B_s} \mid B_r, 0 \le r \le s\right) \\
= \;& \mathrm{E}\left(G_{t-s}\right) e^{\mu s + \sigma B_s} \\
= \;& G_0 e^{(t-s)(\mu + \sigma^2/2)} e^{\mu s + \sigma B_s} \\
= \;& G_s e^{(t-s)(\mu + \sigma^2/2)}
\end{aligned}$$

- We see that G_t is a martingale with respect to B_t if and only if $\mu + \sigma^2/2 = 0$.

The Black-Scholes formula for option pricing

- It is not easy to get a reliable estimate for μ in the model of a stock, even if one can get an estimate of σ, the volatility.
- A possibility is to *assume* that the *discounted* value of the stock is a martingale relative to Brownian motion: So on average it is not better or worse to invest in the stock than in a "risk free" investment.
- This means that $\mu - r + \sigma^2/2 = 0$, i.e., $\mu = r - \sigma^2/2$.
- Plugging this into the formula for the value of a stock option and multiplying with e^{-rt} we get

  $$e^{-rt}\,\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - e^{-rt} K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

  where $\beta = (\log(K/G_0) - (r - \sigma^2/2)t)/\sigma$.
- This is the Black-Scholes formula for option pricing.
- With $r = 0.02$, $G_0 = 67.3$, $\sigma = 0.3$, $t = 3$, and $K = 70$, we get the discounted stock option price 3.39.