MVE550 2022 Lecture 14
Remaining part of Dobrow Chapter 8
Review. Where to go from here.
Exam tips

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Review: Modelling stock prices with geometric Brownian motion

- The stochastic process

$$G_t = G_0 e^{\mu t + \sigma B_t}$$

  where B_t is Brownial motion is called *geometric Brownian motion* with drift parameter μ and variance parameter σ^2.
- Fixing parameters G_0 (start price) μ (trend) and σ (volatility) gives a model for a stock price.
- The option of buying the stock at future time t for price K has value

$$\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 e^{t(\mu+\sigma^2/2)} \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

  where $\beta = (\log(K/G_0) - \mu t)/\sigma$.
- If r is the "risk free" rate of return, the return on the stock investment may be modelled with

$$e^{-rt} G_t = e^{-rt} G_0 e^{\mu t + \sigma B_t} = G_0 e^{(\mu - r)t + \sigma B_t}$$

Martingales

- A stochastic process $(Y_t)_{t \geq 0}$ is a *martingale* if for $t \geq 0$
  - $\mathrm{E}(Y_t \mid Y_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) < \infty$.
- Brownian motion is a martingale.
- $(Y_t)_{t \geq 0}$ is a *martingale with respect to* $(X_t)_{t \geq 0}$ if for all $t \geq 0$
  - $\mathrm{E}(Y_t \mid X_r, 0 \leq r \leq s) = Y_s$ for $0 \leq s \leq t$.
  - $\mathrm{E}(|Y_t|) < \infty$.
- Example: Define $Y_t = B_t^2 - t$ for $t \geq 0$. Then Y_t is a martingale with respect to Brownian motion.

Geometric Brownian motion can be a martingale

Let G_t be Geometric Brownian motion. We get

$$\begin{aligned}
& \mathrm{E}\left(G_t \mid B_r, 0 \le r \le s\right) \\
= \quad & \mathrm{E}\left(G_0 e^{\mu t + \sigma B_t} \mid B_r, 0 \le r \le s\right) \\
= \quad & \mathrm{E}\left(G_0 e^{\mu(t-s) + \sigma(B_t - B_s)} e^{\mu s + \sigma B_s} \mid B_r, 0 \le r \le s\right) \\
= \quad & \mathrm{E}\left(G_{t-s}\right) e^{\mu s + \sigma B_s} \\
= \quad & G_0 e^{(t-s)(\mu + \sigma^2/2)} e^{\mu s + \sigma B_s} \\
= \quad & G_s e^{(t-s)(\mu + \sigma^2/2)}
\end{aligned}$$

- We see that G_t is a martingale with respect to B_t if and only if $\mu + \sigma^2/2 = 0$.

The Black-Scholes formula for option pricing

- It is not easy to get a reliable estimate for μ in the model of a stock, even if one can get an estimate of σ, the volatility.
- A possibility is to *assume* that the *discounted* value of the stock is a martingale relative to Brownian motion: So on average it is not better or worse to invest in the stock than in a "risk free" investment.
- This means that $\mu - r + \sigma^2/2 = 0$, i.e., $\mu = r - \sigma^2/2$.
- Plugging this into the formula for the value of a stock option and multiplying with e^{-rt} we get

  $$e^{-rt}\,\mathrm{E}\left(\max\left(G_t - K, 0\right)\right) = G_0 \Pr\left(B_1 > \frac{\beta - \sigma t}{\sqrt{t}}\right) - e^{-rt} K \Pr\left(B_1 > \frac{\beta}{\sqrt{t}}\right)$$

  where $\beta = (\log(K/G_0) - (r - \sigma^2/2)t)/\sigma$.
- This is the Black-Scholes formula for option pricing.
- With $r = 0.02$, $G_0 = 67.3$, $\sigma = 0.3$, $t = 3$, and $K = 70$, we get the discounted stock option price 3.39.

Review: Stochastic processes
Dobrow

- Chapters 2, 3: Discrete time discrete state space Markov chains.
- Chapter 4: Branching processes.
- Chapter 5: MCMC
- Chapter 6: Poisson processes.
- Chapter 7: Continuous time discrete state space Markov chains.
- Chapter 8: Brownian motion.

Review: Introduction to Bayesian inference
Compendium

- Basic ideas of Bayesian inference.
- The idea of conjugacy and how to use it in computations.
- Discretization and computation of integrals used in low dimensions.
- The idea and usage of Hidden Markov Models.
- Using MCMC for Bayesian inference. Gibbs sampling.
- Specifically we looked at Bayesian inference for:
  - Discrete time discrete state space Markov chains.
  - Hidden Markov models (HMM).
  - Branching processes.
  - Poisson processes (also spatial).

Going forward: Stochastic processes

- Infinite collections of random variables. We have only looked at a few examples.
- For example Brownian motion: An entire PhD level course exclusively about Brownian motion was given last year by Jeff Steif.
- A number of proofs actually need measure theory, which is not covered in this course.
- For those with an interest, I strongly recommend Chapter 9 of Dobrow, introducing Stochastic Calculus!
- Some related courses of possible interest:
  - MVE170 / MSG800 Basic Stochastic Processes, given by Patrik Albin in reading period 2.
  - MVE140 / MSA150 Foundations of Probability Theory, given by Sergei Zuev in reading period 2.
  - TMS165 / MSA350 Stochastic Analysis, given by Patrik Albin in reading period 1.
  - TMV100 / MMA100 Integration Theory, given by Jeff Steif in reading period 1.

Going forward: Stochastic modelling and stochastic differential equations

- Some more advanced courses that may not be given every year: MVE330 Stochastic Processes; MMA630 Computational methods for SDEs; MSF600 Advanced topics in probability. A course in Brownian Motion (Jeff Steif).
- Generally, stochastic modelling means probabilistic modelling of real phenomena, whether the model is a stochastic process or a finite collection of random variables.
- A large number of application areas: Basically any system with uncertainty.
- Stochastic Differential Equations (SDE) and Stochastic Partial Differential Equations (SPDE), see Dobrow Chapter 9 for an introduction. Active area at Chalmers Mathematical Sciences.

Going forward: Bayesian inference

- Stochastic models to be used for real applications almost always need fitting (inference) of their parameters using data.
- In Bayesian inference, we specify an entire stochastic model, also for the parameters (using a prior). Then we use for prediction the conditional distribution given the observed data.
- In practice, the main difference in most cases to frequentist inference is that Bayesian inference averages over a posterior for the parameters instead of using a single parameter estimate.
- In this course, we have looked at Bayesian inference for
  - Small toy models
  - Discrete-time discrete state space Markov chains, and Hidden Markov Models (HMM).
  - Branching processes
  - In assignments: Poisson processes and Continuous-time discrete state space Markov chains.
- We have also looked at how to use Markov chains (MCMC) for Bayesian inference.

Going forward: Bayesian inference

- The intentions in this course:
  - To give a small introduction to Bayesian thinking.
  - Exemplify some simple inference tools for our stochastic processes.
- Bayesian inference is part of many later statistics courses, but I would like to advertise my own course, MVE187 / MSA101 Computational methods for Bayesian inference, in reading period 1. Some subjects covered:
  - More on conjugacy and simple computations.
  - Much more on MCMC.
  - Hamiltonian MCMC.
  - Information theory and the EM algorithm.
  - State space models.
  - Graphical models.
  - Variational Bayes.

Studying for the exam

- Make sure you have *some* general understanding of *all* parts of the course.
- Make sure you have tried out and played around with the small R codes on Canvas. For me, trying out actual computation of things is very helpful to make them more concrete.
- You don't have to memorise proofs but going through them can be a great way to increase understanding.
- DO OLD EXAM QUESTIONS! Note how some, during the pandemic, had "all aids allowed", while you will be allowed a "Chalmers approved calculator".

General exam tips

- Write clearly, and *precisely*, with full sentences. If I'm not sure what you mean, I will *not* necessarily make the interpretation that is most in your interest.
- As long as you are clear and cover everything you want to say, writing short is better than writing long.
- Make sure you answer all parts of a question, and answer exactly what is asked for!
- Make sure you attempt to answer all questions: I cannot give you any points for a question if you do not answer anything.
- If you have *any* doubts about the interpretation of a question, do not hesitate to ask me! I plan to visit the exam around 9:30 and 11:30. The exam personnel may also contact me by phone.
- Distribute your time wisely.