8.11

a) $T_{\mu\nu} F^{\mu\nu} \overset{\text{obvious}}{=} \frac{1}{2}\left(T_{\mu\nu} F^{\mu\nu} + T_{\mu\nu} F^{\mu\nu}\right) \overset{\mu,\nu \text{ are dummy indices}}{=}$

$= \frac{1}{2}\left(T_{\mu\nu} F^{\mu\nu} + T_{\nu\mu} F^{\nu\mu}\right) \overset{\text{use symmetry/antisym.}}{=}$

$= \frac{1}{2}\left(T_{\mu\nu} F^{\mu\nu} + T_{\mu\nu}\left(-F^{\mu\nu}\right)\right) =$

$= \frac{1}{2}\left(T_{\mu\nu} F^{\mu\nu} - T_{\mu\nu} F^{\mu\nu}\right) = 0.$

Equivalently, you can just say $T_{\mu\nu} F^{\mu\nu} \underset{(\text{dummy})}{=} T_{\nu\mu} F^{\nu\mu} \underset{(\text{symmetry})}{=}$

$= T_{\mu\nu}\left(-F^{\mu\nu}\right) = -T_{\mu\nu} F^{\mu\nu} \Rightarrow T_{\mu\nu} F^{\mu\nu} = 0.$

b) I need to show that $X^{\nu\mu} = F^{\nu\rho} F_{\rho}{}^{\mu}$ is symmetric then I can use a).

$$X^{\mu\nu} = F^{\mu\rho} F_\rho{}^\mu = -F^{\rho\mu} F_\rho{}^\nu =$$

$$= -F_\rho{}^\mu F^{\rho\nu} = -F_\rho{}^\mu (-F^{\nu\rho}) =$$

$$= +F_\rho{}^\mu F^{\nu\rho} = F^{\nu\rho} F_\rho{}^\mu = X^{\nu\mu}$$

Thus: $F_{\mu\nu} X^{\mu\nu} = 0$.

c) I need to show that

$Y^{\nu\mu} = F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}$ is antisymmetric

then I can use a).

$$Y^{\mu\nu} = F^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} = // \text{dummy indices} //$$

$$= F^{\mu\sigma} F_{\sigma\rho} F^{\rho\nu} = F^{\rho\nu} F_{\sigma\rho} F^{\mu\sigma}$$

$$= // \text{antisymm.} // (-F^{\nu\rho})(-F_{\rho\sigma})(-F^{\sigma\mu})$$

$$= -F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} = -Y^{\nu\mu}$$

Thus $T_{\mu\nu} Y^{\mu\nu} = 0$.