Polynomials as Vectors

Jimmy Aronsson

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1 Introduction

When hearing the word *vector*, many people picture an arrow; a directed magnitude. The *velocity vector* of a moving car, for instance, is an arrow pointing in the current direction of travel, and the length of this vector determines the speed; a longer velocity vector means a higher speed.

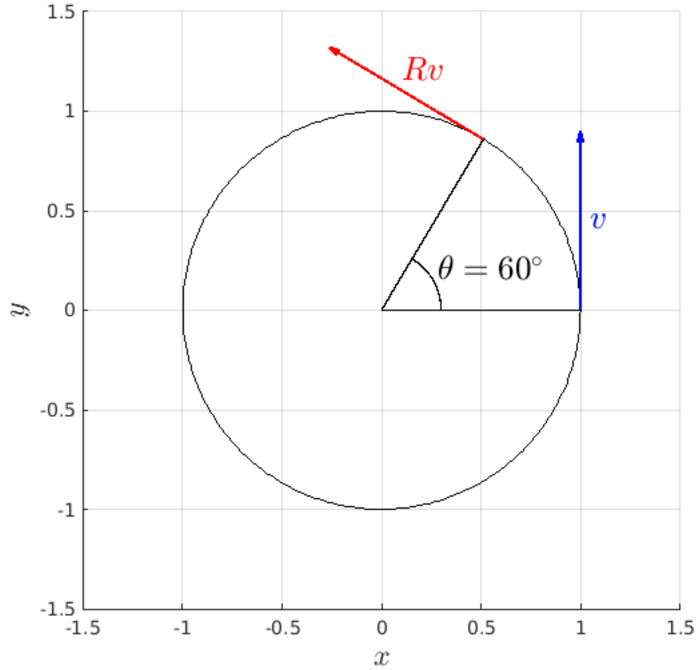
Those who have studied linear algebra also know how these arrow vectors can be represented as columns, lists of numbers that specify how far an arrow points in each direction:

$$v = \begin{bmatrix} x \\ y \end{bmatrix}.$$

You also know how matrices can be used to transform vectors:

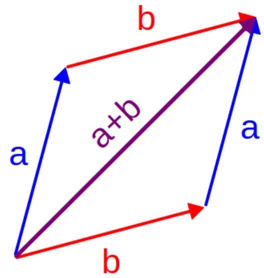
$$\underbrace{\begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} x\cos(60°) - y\sin(60°) \\ x\sin(60°) + y\cos(60°) \end{bmatrix}}_{Rv} = \frac{1}{2}\begin{bmatrix} x - \sqrt{3}y \\ \sqrt{3}x + y \end{bmatrix}.$$

This particular matrix R performs a $60°$ counter-clockwise rotation of any 2D vector:



Mathematicians use a more abstract definition of vectors. Rather than going through the full definition, let me just say that a *vector space* is a collection of objects that can 1. be combined into other objects of the same type, and 2. be multiplied by real numbers to scale them, and the objects in such a collection are then called *vectors*. Importantly, abstract vectors do not need to have a clear geometric interpretation, although they may.

The arrows we discussed above satisfy this definition, because any two arrows can be added to form a new arrow,



and multiplying an arrow by some real number changes its length by a proportionate amount. In this sense, the collection of all such arrows satisfy the definition of a vector space, hence the arrows themselves can be regarded as vectors (objects that form a vector space).

There are many other things that also satisfy this abstract definition of a vector space. One such example of vectors is: Polynomials.

2 Polynomials

Instead of studying the collection of *all* possible polynomials of arbitrary large degree, which is a very large collection, let us instead study polynomials *of a particular degree*. For example, the collection of all degree 3 polynomials is denoted

$$P_3 = \left\{a_0 + a_1x + a_2x^2 + a_3x^3 \;\middle|\, a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

Similarly, for any natural number $n = 0, 1, 2, 3, \ldots$, the collection of all degree n polynomials is

$$P_n = \left\{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \;\middle|\, a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\right\}.$$

Any degree 2 polynomial $p(x) = a_0 + a_1x + a_2x^2$ can be thought of as a degree 3 polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + 0x^3$$

whose last coefficient $a_3 = 0$. The collections therefore contain each other:

$$P_0 \subset P_1 \subset P_2 \subset P_3 \subset \cdots$$

From now on, let's focus on degree 3 polynomials to keep the number of terms manageable.

In order for the collection P_3 to satisfy the definition a vector space, we need some way to combine two arbitrary polynomials of degree 3 into a new polynomial of degree 3, and some way to multiply polynomials of degree 3 by real numbers. There are natural ways to do both of these things: First, given any two polynomials of degree 3, such as

$$p(x) = x^3 + 3x - 4, \quad \text{and} \quad q(x) = 0.2x^3 - x^2 + 1,$$

we simply compute their sum,

$$\begin{aligned}(p+q)(x) &= (x^3+3x-4)+(0.2x^3-x^2+1) = \\ &= 1.2x^3-x^2+3x-3,\end{aligned}$$

and observe that we obtain another polynomial of degree 3. So the operation that combines two polynomials to form a new polynomial of the same degree, is simply to compute their sum.

Second, multiplication by real numbers also produce degree 3 polynomials:

$$10 * q(x) = 10 * (0.2x^3 - x^2 + 1) = 2x^3 - 10x^2 + 10.$$

The collection P_3 therefore satisfies the aforementioned definition of an abstract vector space, and the objects it contain (i.e. polynomials of degree 3) can rightfully be called vectors. Naturally, there is nothing special about degree 3 polynomials. It is equally true that, say, P_2, P_5 and in general P_n forms a vector space.

3 Basis vectors

Observe that the general expression of a degree 3 polynomial,

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3, \tag{1}$$

can be interpreted as a linear combination of the polynomials 1, x, x^2, and x^3. Any polynomial of degree 3 is uniquely characterized by the specific coordinates a_0, a_1, a_2, a_3 used in the linear combination (1), which means that 1, x, x^2, and x^3 forms a *basis* of P_3. It also allows us to stop writing out the x's and just represent each polynomial by a list of coordinates a_k:

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{2}$$

Because we need 4 numbers a_0, a_1, a_2, a_3 to specify any degree 3 polynomial - equivalently, because there are 4 different basis vectors 1, x, x^2, and x^3, the vector space P_3 has dimension 4.

More generally, the vector space P_n of polynomials with degree n has dimension $n+1$, because you need $n+1$ different numbers $a_0, a_1, a_2, \ldots, a_n$ to specify any such polynomial - equivalently, because P_n has a basis consisting of the vectors $1, x, x^2, \ldots, x^n$.

4 Differentiation

It's an interesting fact that the very much analytic process of differentiation (and, as we shall see later, integration) of polynomials is a linear transformation between vector spaces; If we write any polynomial of degree 3 as a column vector like in equation (2), we can compute its derivative (which is a degree 2 polynomial) using matrix multiplication. To show this, first recall that the derivative of x^2 is $2x$, the derivative of x^3 is $3x^2$, and so on. In general,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^{k-1}. \tag{3}$$

The derivative of any polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ can then be obtained by computing the derivative of each x^k term individually, using equation (3):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}p(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \\
&= a_0\left(\frac{\mathrm{d}}{\mathrm{d}x}1\right) + a_1\left(\frac{\mathrm{d}}{\mathrm{d}x}x\right) + a_2\left(\frac{\mathrm{d}}{\mathrm{d}x}x^2\right) + a_3\left(\frac{\mathrm{d}}{\mathrm{d}x}x^3\right) = \\
&= a_0 * 0 + a_1 * 1 + a_2 * 2x + a_3 * 3x^2 = \\
&= a_1 + 2a_2x + 3a_3x^2.
\end{aligned}$$

Differentiating a polynomial always reduces its degree by 1, as illustrated in the above example where the general degree 3 polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ turned into the degree 2 polynomial $p'(x) = a_1 + 2a_2x + 3a_3x^2$. This means that the differentiation process is a function

$$\frac{\mathrm{d}}{\mathrm{d}x} : P_3 \to P_2.$$

This function is also linear, because you compute derivatives term by term. In other words, the differentiation process is a *linear transformation* that turns degree 3 polynomials into degree 2 polynomials. In fact, we can write down its matrix by examining how it changes column vectors: If an arbitrary degree 3 polynomial is written as a column vector

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in P_3,$$

then we know its derivative is the column vector

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix} \in P_2,$$

and the matrix that maps $p(x)$ to its derivative $p'(x)$ can be written as the differentiation matrix

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Indeed,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0a_0 + 1a_1 + 0a_2 + 0a_3 \\ 0a_0 + 0a_1 + 2a_2 + 0a_3 \\ 0a_0 + 0a_1 + 0a_2 + 3a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}.$$

Interestingly, the differentiation matrix D is not invertible, which reflects something we know from analysis: Differentiation cannot be fully undone; you cannot fully recover any polynomial $p(x)$ by only knowing its derivative $p'(x)$, because the constant a_0 disappears in the process.

More generally, differentiation of degree n polynomials is also a linear transformation

$$\frac{\mathrm{d}}{\mathrm{d}x} : P_n \to P_{n-1},$$

and one can write down its $(n-1) \times n$ matrix.

5 Integration

Just like differentiation is a linear transformation

$$\frac{\mathrm{d}}{\mathrm{d}x} : P_3 \to P_2$$

that reduces the degree of a polynomial by 1, integration is a linear transformation

$$\int \mathrm{d}x : P_2 \to P_3$$

that instead *increases* the degree of a polynomial by 1. To see this, first recall that the (indefinite) integral of each basis element x^k is given by

$$\int x^k \ \mathrm{d}x = \frac{1}{k+1} x^{k+1},$$

where we ignore the additive constant $+C$ for simplicity by setting $C = 0$. Integration is then defined for arbitrary polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \in P_2$$

by integrating each term separately:

$$\begin{aligned}
\int p(x) \ \mathrm{d}x = \int a_0 + a_1 x + a_2 x^2 \ \mathrm{d}x &= \\
&= a_0 \left(\int 1 \ \mathrm{d}x \right) + a_1 \left(\int x \ \mathrm{d}x \right) + a_2 \left(\int x^2 \ \mathrm{d}x \right) = \\
&= a_0(x) + a_1 \left(\frac{1}{2} x^2 \right) + a_2 \left(\frac{1}{3} x^3 \right) = \\
&= a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 = \begin{bmatrix} 0 \\ a_0 \\ a_1/2 \\ a_2/3 \end{bmatrix} \in P_3.
\end{aligned}$$

This "integrate term by term" process precisely means that integration is a linear transformation, hence it can be written in terms of column vectors and matrix multiplication:

$$\int p(x) \ \mathrm{d}x = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 = \begin{bmatrix} 0 \\ a_0 \\ a_1/2 \\ a_2/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

In other words, integration of any degree 2 polynomial is performed by the integration matrix

$$I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

While differentiation is not invertible (because you lose the constant a_0), integration is invertible; if you integrate a polynomial $p(x)$, you can then differentiate the resulting polynomial $\int p(x)\, \mathrm{d}x$ to get back the same polynomial $p(x)$ that you started with. In terms of matrices, differentiation followed by integration produces a linear transformation $ID : P_3 \to P_3$ given by

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{D} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{ID} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{4}$$

confirming our previous claim: First differentiating (D) and then integrating (I) a polynomial doesn't give us back the original polynomial, because we lose the constant a_0.

In contrast, first integrating and then differentiating a polynomial produces a linear transformation $DI : P_2 \to P_2$ which can be written in terms of matrices as

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}_{I} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{DI} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \tag{5}$$

and which *does* give us back the polynomial we started with.

Not only does this example show an interesting relation between analysis and linear algebra, namely that highly analytic processes such as differentiation and integration can be thought of as linear transformations, and (at least in the case of polynomials) be written on matrix form. It also shows that certain matrices, such as the differentiation matrix D, can have a *right-inverse* without having a *left-inverse*. Indeed, equation (5) shows that multiplying D from the right by the integration matrix I produces the 3×3 identity matrix

$$DI = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

but equation (4) shows that multiplying D from the *left* by I *doesn't* produce the 4×4 identity matrix:

$$ID = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Again, this reflects the fact that integration can be undone by differentiating, while differentiation cannot be undone by integrating, since we lose all information about the constant a_0.