MATHEMATICS
University of Gothenburg and Chalmers University of Technology.
Examination in algebra : MMG500 and MVE 150, 2019-03-22.
No aids are allowed. Telephone 031-772 5325.

1) Let R be an integral domain with four elements $0,1,a$ and b where 1 is the neutral element for multiplication. Prove the following statements in R.

a) $1+1=0$. 2p

b) $a+1=b$. 1p

c) $a^2=b$. 2p

2) Let R be the set of all rational numbers of the form $m/2^n$ for integers m and n.

a) Show that R is a subring of $\mathbf{Q}$. 2p

b) Which of the integers 2, 4 and 6 are irreducible in R ? 2p

Which of these three integers are units in R ?

3a) Determine the number of elements of order 5 in S_5. 2p

3b) Determine the number of subgroups of order 5 in S_5. 2p

4) Let T be the set of subgroups of order 5 of S_5. If $\sigma\in S_5$, let $\pi_\sigma\colon T\to T$ be the bijective map which sends a subgroup H of order 5 to the subgroup $\sigma H\sigma^{-1}:=\{\sigma h\sigma^{-1} : h\in H\}$ in T.

a) Prove that the map $\pi\colon S_5\to \mathrm{Sym}(T)$, which sends $\sigma\in S_5$ to $\pi_\sigma\in \mathrm{Sym}(T)$ gives an action of S_5 on T. 2p

b) The above action of S_5 on T is transitive by a theorem of Sylow. Use this to show that π is injective. 2p

5a) Show that each group has at most one neutral element. 2p

5b) Show that each element of a group has at most one inverse. 2p

6) Prove that congruence modulo n is an equivalence relation on $\mathbf{Z}$ for each positive integer n. 4p

You may use the theorems in Durbin's book to solve the first 4 exercises.
But all claims should be motivated !