MVE035/600, VT-20: IMPLICIT FUNCTION THEOREM FOR MORE THAN ONE EQUATION

Let n, k be positive integers and let $F_1, \dots, F_k$ be functions of $n+k$ variables $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}$. Consider a system of equations, i.e.: of level surfaces,

$$F_1(x_1, \dots, x_{n+k}) = c_1, \\ \vdots \\ F_k(x_1, \dots, x_{n+k}) = c_k. \tag{0.1}$$

The basic question is: when can we eliminate the k variables $x_{n+1}, \dots, x_{n+k}$ from this system ?

Suppose each F_i is a C^1-function and, for the sake of argument, suppose we can indeed eliminate the above k variables so that there are implicitly defined C^1-functions $f_1, \dots, f_k$ of n variables such that

$$x_{n+m} = f_m(x_1, \dots, x_n), \quad m = 1, \dots, n. \tag{0.2}$$

Substituting (0.2) into (0.1), the i:th equation, for each $i = 1, \dots, k$, reads

$$F_i(x_1, \dots, x_n, f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) = c_i. \tag{0.3}$$

Using the chain rule, we can partially differentiate (0.3) with respect to x_j, for each $j = 1, \dots, n$, and find that

$$\frac{\partial F_i}{\partial x_j} + \sum_{m=1}^{k} \frac{\partial F_i}{\partial x_{n+m}} \frac{\partial f_m}{\partial x_j} = 0. \tag{0.4}$$

Note that (0.4) describes in total a system of kn equations, one for each $i = 1, \dots, k$ and $j = 1, \dots, n$. If you stare long enough, you'll see that this system can be written in matrix form as

$$B + AX = 0, \tag{0.5}$$

where $A = (a_{uv})$ is a $(k \times k)$-matrix, $B = (b_{uv})$ is a $(k \times n)$-matrix and $X = (c_{uv})$ is a $(k \times n)$-matrix. Specifically,

$$a_{uv} = \frac{\partial F_u}{\partial x_{n+v}}, \quad b_{uv} = \frac{\partial F_u}{\partial x_v}, \quad c_{uv} = \frac{\partial f_u}{\partial x_v}. \tag{0.6}$$

Note that the only *unknowns* here are the functions f_m, $m = 1, \dots, k$, hence X is the "unknown" matrix in (0.5). Eq. (0.5) has a unique solution if and only if the matrix A is invertible, hence if and only if $\det(A) \neq 0$.

The Implicit Function Theorem states that, if $\det(A) \neq 0$ in a point $\boldsymbol{a} \in \mathbb{R}^{n+k}$ satisfying the system (0.1), then indeed there exist C^1-functions f_m satisfying (0.2) in a neighborhood of that point. Moroever the kn partial derivatives $\frac{\partial f_u}{\partial x_v}$, $u = 1, \dots, k$, $v = 1, \dots, n$, which are the entries of the matrix X, are given in matrix form by $X = -A^{-1}B$.

This is the most general form of the IFT. It reduces to what we presented in class when $k = 1$.