K14

$$dS = r^2 \sin\phi \, d\phi \, d\theta = \{r=2\} = 4\sin\phi \, d\phi \, d\theta$$

Integrand: $(x^2+y^2)z = (4-z^2)z = (4-4\sin^2\phi)2\sin\phi = 8\sin\phi\underbrace{(1-\sin^2\phi)}_{\cos^2\phi}$

$$\iint_S (x^2+y^2)z\,dS = \int_0^{2\pi}\int_0^{\pi/2} 32\sin^2\phi(1-\sin^2\phi)\,d\phi\,d\theta = 32\int_0^{\pi/2}\sin^2\phi(1-\sin^2\phi)[\theta]_0^{2\pi}\,d\phi =$$

$$= 64\pi\int_0^{\pi/2}(\sin^2\phi - \sin^4)\,d\phi = (*)$$

Dubbla vinkeln: $\cos(2u) = 1 - 2\sin^2 u$

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\Rightarrow \int \sin^2 u\,du = \int \frac{1-\cos 2u}{2}\,du = \frac{u + \frac{1}{2}\sin 2u}{2}$$

$$\sin^4 u = \frac{1 - 2\cos 2u + \cos^2 2u}{4}$$

Dubbla vinkeln: $\cos(2u) = 2\cos^2 u - 1$

$$\cos^2 2u = \frac{\cos 4u + 1}{2}$$

$$\sin^4 u = \frac{2 - 4\cos 2u + \cos 4u + 1}{8} =$$

$$= \frac{3 - 4\cos 2u + \cos 4u}{8}$$

$$\Rightarrow \int \sin^4 u\,du = \int \frac{3 - 4\cos 2u + \cos 4u}{8}\,du = \frac{3u + 2\sin 2u - \frac{1}{4}\sin 4u}{8}$$

$$\Rightarrow (*) = 64\pi\int_0^{\pi/2}(\sin^2\phi - \sin^4\phi)\,d\phi = \left[\frac{u}{2} + \cancel{\frac{\sin 2u}{4}} - \frac{3u}{8} - \cancel{\frac{\sin 2u}{4}} + \frac{\sin 4u}{32}\right]_0^{\pi/2} =$$

$$= \left[\frac{u}{8} + \frac{\sin 4u}{32}\right]_0^{\pi/2} = \frac{\pi}{16} + \frac{\sin(2\pi) - \sin(0)}{32} = \frac{\pi}{16}$$

K15

(*) = flödet ut ur paraboolen

(**) = flödet in genom bottenytan

Flödet ut = (*) − (**)

(**): $z=0$, $d\mathbf{S} = \hat{\mathbf{k}}\,dxdy$

$$\iint_{\text{skiva}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\text{skiva}} z^2\,dxdy = \{z=0\} = 0$$

(*): $z = 4-x^2-y^2$

$$d\mathbf{S} = (2x\hat{\imath} + 2y\hat{\jmath} + \hat{\mathbf{k}})\,dxdy$$

$$\iint_{\text{parabol}} (2x^3+2y^3+z^2)\,dxdy = \{z=4-x^2-y^2\} = \iint_{\text{parabol}} \left(2x^3+2y^3+(4-x^2-y^2)^2\right)dxdy =$$

$$= \{\text{polära koordinater}\} = \int_0^{2\pi}\int_0^2 \left(2r^4(\cos^3\theta+\sin^3\theta)+r(4-r^2)^2\right)drd\theta =$$

$$= \int_0^{2\pi}\int_0^2 \left(2r(\cos^3\theta+\sin^3\theta)+16r-8r^3+r^5\right)drd\theta =$$

$$= \int_0^{2\pi}\left[\frac{2r^5}{5}(\cos^3\theta+\sin^3\theta)+8r^2-2r^4+\frac{r^6}{6}\right]_0^2 d\theta =$$

$$= \int_0^{2\pi}\left(\frac{64}{5}(\cos^3\theta+\sin^3\theta)+\cancel{32}-\cancel{32}+\frac{32}{3}\right)d\theta = \left\{\begin{matrix}\text{Både } \cos^3\theta \text{ och } \sin^3\theta \\ \text{ger 0 av symmetriskäl} \\ \text{över ett helt varv.}\end{matrix}\right\} =$$

$$= \left[\left(\frac{32}{3}\right)\theta\right]_0^{2\pi} = \frac{64}{3}\pi$$

$$\text{Flöde ut} = (*)-(**) = \frac{64}{3}\pi$$