3rd Lecture: 20/1

Definition 3.1. Let $k \in \mathbb{Z}_+$. A sequence $(a_n)_{n=0}^{\infty}$ of complex numbers is said to satisfy a *recurrance relation of order* k if there exists a function $f : \mathbb{C}^k \to \mathbb{C}$ such that, for all $n \in \mathbb{N}$,

$$a_{n+k} = f(a_{n+k-1},\, a_{n+k-2},\, \dots,\, a_n). \tag{3.1}$$

Observe that a sequence satisfying (3.1) is completely determined by the values $a_0,\, a_1,\, \dots,\, a_{k-1}$.

Remark 3.2. It will sometimes be convenient to index sequences from a different starting point than $n = 0$. We will adjust the starting index in the text below as the situation demands without further comment.

Two basic issues arise in the study of recursively defined sequences:

The Combinatorial Problem. The function f may not be given, and the first step is to find it. This usually involves some kind of "combinatorial reasoning".

The Algebraic Problem. If possible, solve the recurrence (3.1) to find an "explicit formula" for a_n as a function of n. This is a special case of the *computational problem* of computing the elements of the sequence as efficiently as possible. The recursion itself leads to relatively efficient computation - just write a program with a loop. But by an "explicit formula" we have in mind something even better. Moreover, for many applications it may suffice to be able to make a good estimate of a_n for large n, and an explicit formula may give this insight directly, requiring only minimal computation.

Here is a very simple example to illustrate:

Example 3.3. Let a_n denote the number of subsets of an n-element set, say $\{1,\, 2,\, \dots,\, n\}$ WLOG. Clearly $a_0 = 1$, since only the empty subset occurs. Moreover, $a_{n+1} = 2a_n$. For among the subsets of $\{1,\, 2,\, \dots,\, n+1\}$ we can distinguish two types: those which contain $n+1$ and those which don't. Those of each type are in 1-1 correspondence with subsets of $\{1,\, 2,\, \dots,\, n\}$, hence the recursion.

So the sequence $(a_n)_{n=0}^{\infty}$ is defined recursively by

$$a_0 = 1, \quad a_{n+1} = f(a_n), \quad \text{where } f(x) = 2x.$$

Here the algebraic problem is very simple, we can immediately see that $a_n = 2^n$. Note that this explicit formula immediately tells us the order of magnitude of a_n for any particular n, namely $2^n = 10^{n \log_{10} 2} \approx 10^{0.3\, n}$.

Just as, in general, very few algebraic equations can be solved exactly, there are very few functions f for which (3.1) can be solved to yield an explicit formula for a_n. Fortunately, those for which an exact solution procedure exists are also those which tend to arise most commonly. We will in this course focus mostly[1] on *linear* recurrences.

[1] though not entirely, see the discussion of Catalan numbers in Lecture x.

Definition 3.4. The sequence $(a_n)_{n=0}^{\infty}$ is said to satisfy a *homogeneous linear recurrence (HLR)* if the function f in (3.1) is homogeneous and linear, by which we mean that there exist constants $c_1, c_2, \dots, c_k$ such that $f(x_1, x_2, \dots, x_k) = \sum_{i=1}^{k} c_i x_i$.

Homogeneous linear recurrences can always be solved explicitly, modulo solving polynomial equations. Before presenting the formal result we will work through an example and show how things work. Note that Example 3.3 was already a simple example of a HLR.

Example 3.5. Let a_n denote the number of n-bit binary strings which don't contain any consecutive zeroes. We can first check directly that

$a_0 = 1$: the empty string
$a_1 = 2$: 0 or 1
$a_2 = 3$: can't have 00
$a_3 = 5$: can't have 000, 001 or 100.

Already at this stage one might guess that

$$\forall\, n \in \mathbb{N}: \quad a_{n+2} = a_{n+1} + a_n. \tag{3.2}$$

The "combinatorial problem" is to prove this. We do so by considering a sequence of length $n+2$ and two cases:

Case 1: It starts with a 0. Then the second bit must be 1. The remaining n bits satisfy no other conditions than those initially posed, namely there can't be conscutive zeroes. Hence there are a_n possibilities for the remaining bits.

Case 2: It starts with a 1. Then there is no extra condition on the second bit, and the remaining $n+1$ bits satisfy the same conditions as at the outset. There are thus a_{n+1} possible sequences.

Since Cases 1 and 2 are obviously disjoint and exhaust all possibilities, we have by AP proven (3.2).

We now turn to the "algebraic problem" of finding an explicit formula for a_n. Formally, let

$$l^{\infty} := \{(x_n)_{n=0}^{\infty} : x_n \in \mathbb{C}\}.$$

In words, l^{∞} is the vector space of all infinite sequences of complex numbers[2]. It is obviously an infinite-dimensional vector space (in fact, the dimension is uncountable - can you prove this ?). Now let

$$V = \{(x_n)_{n=0}^{\infty} \in l^{\infty} : x_{n+2} = x_{n+1} + x_n \;\forall\, n \in \mathbb{N}\}. \tag{3.3}$$

Firstly, I claim that V is a subspace of l^{∞} - indeed this is equivalent to the fact that the recurrence is a HLR. The proof is simple, but let me give it for the sake of completeness.

[2]The notation is standard in functional analysis.

To show that V is a subspace, we must show that it is closed under vector addition and scalar multiplication.

Closure under addition: Let $\boldsymbol{x} = (x_n)$ and $\boldsymbol{y} = (y_n)$ be elements of V, thus $x_{n+2} = x_{n+1} + x_n$ and $y_{n+2} = y_{n+1} + y_n$. Let $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. Then

$$\begin{aligned} z_{n+2} &\overset{\text{def}}{=} x_{n+2} + y_{n+2} = (x_{n+1} + x_n) + (y_{n+1} + y_n) = \\ &= (x_{n+1} + y_{n+1}) + (x_n + y_n) \overset{\text{def}}{=} z_{n+1} + z_n, \quad \text{v.s.v.} \end{aligned}$$

Closure under scalar multiplication: Let $\boldsymbol{x} = (x_n) \in V$ and $\alpha \in \mathbb{C}$. Let $\boldsymbol{z} := \alpha\boldsymbol{x}$. Then

$$z_{n+2} \overset{\text{def}}{=} \alpha x_{n+2} = \alpha(x_{n+1} + x_n) = (\alpha x_{n+1}) + (\alpha x_n) \overset{\text{def}}{=} z_{n+1} + z_n, \quad \text{v.s.v.}$$

Next, I claim that $\dim(V) = 2$. Intuitively, the reason is that a sequence in V is completely determined by its first two terms. More precisely, there is a vector-space isomorphism $\phi : \mathbb{C}^2 \to V$ given by

$$\phi((a,\, b)) = \text{the unique sequence } (x_n) \in V \text{ such that } x_0 = a,\, x_1 = b.$$

Hence, solving (3.2) reduces to determining a basis for V. At this point one needs to make an "inspired guess", and the right guess is to set $x_n = \alpha^n$ and solve for $\alpha \in \mathbb{C}$. Thus,

$$x_{n+2} = x_{n+1} + x_n \Leftrightarrow \alpha^{n+2} = \alpha^{n+1} + \alpha^n.$$

Now $\alpha \neq 0$ as otherwise the sequence (x_n) would be identically zero and hence not a basis vector. Thus we can cancel α^n from the equation and get an equation which is independent of n:

$$\alpha^2 = \alpha + 1.$$

This is called the *auxiliary/characteristic equation* for the recurrence (3.2). It is a quadratic equation with two roots[3],

$$\alpha_1 = \gamma = \frac{1 + \sqrt{5}}{2}, \quad \alpha_2 = \frac{-1}{\gamma} = \frac{1 - \sqrt{5}}{2}.$$

Hence a basis for V is given by the two sequences $(\gamma^n)_{n=1}^{\infty}$ and $((-1/\gamma)^n)_{n=1}^{\infty}$, since whenever $\alpha_1 \neq \alpha_2$ it is clear that the sequences (α_1^n) and (α_2^n) are linearly independent elements of l^∞.

Thus the sequence $(a_n)_{n=0}^{\infty}$ must be a linear combination of these two, i.e.: there exist constants C_1, C_2 such that

$$a_n = C_1 \cdot \gamma^n + C_2 \cdot \left(\frac{-1}{\gamma}\right)^n.$$

To determine C_1 and C_2 we insert the initial conditions $n = 0$ and $n = 1$:

$$\begin{aligned} n = 0: &\quad a_0 = 1 = C_1 + C_2, \\ n = 1: &\quad a_1 = 2 = C_2 \cdot \gamma + C_2 \cdot \frac{-1}{\gamma}. \end{aligned}$$

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[3] γ is standard notation for the golden ratio.

This is just a system of two linear equations in two unknowns, so a standard Gauss elimination problem, though with somewhat ugly coefficients. You can check that the solution is $C_1 = 3 - \gamma$, $C_2 = \gamma - 2$ and hence that the explicit formula for a_n is

$$a_n = (3 - \gamma) \cdot \gamma^n + (-1)^n \cdot \frac{\gamma - 2}{\gamma^n}. \tag{3.4}$$

Remark 3.6. (Fibonacci numbers). Fibonnaci was apparently interested in studying the reproductive behaviour of rabbits (who are a good choice of species to study for this purpose because their rate of reproduction is unusually fast for mammals). Fibonacci's model makes the following assumptions. Some of them obviously sound stupid on the level of individual rabbits, but in such cases one should instead imagine that they are statements about average behaviour in a large population - see the remark below.

ASSUMPTION 1: Rabbits live forever.
ASSUMPTION 2: Rabbits form monogamous pairs.
ASSUMPTION 3: Rabbit pregnancy lasts one month.
ASSUMPTION 4: Rabbit childhood lasts one month.
ASSUMPTION 5: Adult females conceive new offspring as soon as the last batch have been born.
ASSUMPTION 6: Each conception results in a pair of twins, one of each sex.

Remark: Assumptions 3-6 should be thought of as representing average behaviour, in which case Assumption 2 becomes superfluous. Assumption 1 is, however, a serious restriction and obviously implies that the model is unrealistic over longer time periods. Hoever, one can still consider it as a first step in trying to understand how quickly a population of rabbits will proliferate over shorter time scales.

Suppose we begin with a single pair of newborn rabbits. For each $n \geq 0$, let f_n denote the number of rabbit pairs living after n months. Thus $f_0 = 1$ by assumption. Also, $f_1 = 1$ since after one month the newborn pair will have grown up but not yet produced any offspring. Now I claim that, for all $n \in \mathbb{N}$,

$$f_{n+2} = f_{n+1} + f_n. \tag{3.5}$$

To see this, first write $f_n = v_n + b_n$, where v_n is the number of adult pairs after $n - 1$ months and b_n is the number of newborn pairs at this time. Then observe that

(i) $v_{n+2} = f_{n+1}$ since every rabbit which was alive one month previously will now be an adult,

(ii) $b_{n+2} = f_n$ since only those rabbit pairs which were alive two months previously will have produced offspring in the previous month, since they first needed to become adults the month before.

Then (3.5) follows from (i) and (ii). To summarise, the sequence $(f_n)_{n=0}^{\infty}$ of Fibonacci numbers is defined by the recurrence

$$f_0 = 1, \quad f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n \;\; \forall\, n \in \mathbb{N}. \tag{3.6}$$

Comparing with Example 3.5, we see that $f_n = a_{n-1}$ for all $n \geq 1$. In particular, as $n \to \infty$, since $|-1/\gamma| < 1$ it follows from (3.4) that $f_n = C \cdot \gamma^n + o(1)$, $C = \frac{3-\gamma}{\gamma} = \cdots = \frac{\sqrt{5}(3-\sqrt{5})}{2}$. In particular, the Fibonacci numbers grow exponentially with exponent γ.

Given a HLR of degree k, the corresponding vector space V will be k-dimensional. The auxiliary equation will be a polynomial equation of degree k so, if we're lucky, it will have k distinct roots in the complex numbers[4], which will give us a complete basis of k vectors for V. The only thing that can possibly go wrong is that the auxiliary equation has one or more repeated roots. However, this situation can be handled. Before presenting the general result, we do an example:

Example 3.7. Let's solve the recurrence

$$u_0 = 1, \quad u_1 = 2, \quad u_{n+2} = 6u_{n+1} - 9u_n \ \ \forall\, n \geq 0.$$

The auxiliary equation is $\alpha^2 - 6\alpha + 9 = 0$, which has a repeated root $\alpha_{1,2} = 3$. Hence the sequence (3^n) is one basis vector. It turns out that the second one is given by the sequence $(n \cdot 3^n)$. Hence, there exist constants C_1, C_2 such that

$$u_n = (C_1 + C_2 n)3^n.$$

We insert the initial conditions

$$n = 0: \ \ u_0 = 1 = C_1, \quad n = 1: \ \ 2 = u_1 = 3(C_1 + C_2) \Rightarrow C_2 = -1/3.$$

Hence, $u_n = \left(1 - \frac{n}{3}\right) 3^n$.

Theorem 3.8. *Suppose the sequence $(a_n)_{n=0}^{\infty}$ satisfies the HLR*

$$a_{n+k} = \sum_{i=1}^{k} c_i a_{n+k-i} \quad \forall\, n \geq 0. \tag{3.7}$$

If α is a root of multiplicity l of the auxiliary equation

$$x^k = \sum_{i=1}^{k} c_i x^{k-i} \tag{3.8}$$

then, for any polynomial $p(x) \in \mathbb{C}[x]$ of degree $l-1$, $a_n = p(n) \cdot \alpha^n$ is a solution of (3.7). Hence, if $\alpha_1, \ldots, \alpha_t$ are all the distinct roots of (3.8), with multiplicities $l_1, \ldots, l_t$, then the general solution of (3.7) is given by

$$a_n = \sum_{i=1}^{t} p_i(n) \cdot \alpha_i^n, \tag{3.9}$$

where $p_i(x)$ is a polynomial of degree l_i.

We'll give the proof next time.

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[4]Recall the *Fundamental Theorem of Algebra*, which states that every polynomial with complex coefficients can be completely factorised in $\mathbb{C}$ - the technical terminology being that $\mathbb{C}$ is an algebraically closed field.