1st Lecture: 18/1

The first part of the course is concerned with what is often called *Enumerative Combinatorics* which, informally, is the art of counting. In general terms one is interested in "clever" means of counting the sizes of finite sets, or at least estimating them - one is usually dealing with large sets, or a sequence $(A_n)_{n=1}^{\infty}$ of sets and then interested in estimating $|A_n|$ as $n \to \infty$.

Two very basic principles underlie a very common method of reasoning:

Addition Principle (AP). *Let A and B be two finite, disjoint sets (i.e.: $A \cap B = \phi$). Then*

$$|A \cup B| = |A| + |B|.$$

More generally, if $A_1, A_2, \ldots, A_k$ are finite, pairwise disjoint sets, i.e.: $A_i \cap A_j = \phi$ for all $i \neq j$, then

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|.$$

Recall that the *Cartesian product* of two sets A and B is defined and denoted as

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Multiplication Principle (MP). *Let A and B be two finite sets. Then*

$$|A \times B| = |A| \times |B|.$$

More generally, if $A_1, A_2, \ldots, A_k$ are finite sets, then

$$\left| \prod_{i=1}^{k} A_i \right| = \prod_{i=1}^{k} |A_i|.$$

The above two principles can be reformulated in various ways, each of which provides its own insights. One viewpoint which is particularly useful for counting problems is the following:

Addition Principle (2nd formulation). *Let A and B be two finite, disjoint sets. The number of ways to choose an element from **either** A **or** B is $|A| + |B|$.*

Multiplication Principle (2nd formulation). *Let A and B be two finite, disjoint sets. The number of ways to choose **both** an element from A **and** an element from B is $|A| \times |B|$.*

Choice with/without repitition and with/without order. We want formulas for the number of ways to choose n objects from a set of k distinct objects, under four different conditions, depnding on whether or not repitition is allowed and whether or not the order of choice matters.

Proposition 1.1. *The number of ways to choose k objects from a set of n distinct objects, where repitition is allowed and order of choice matters is n^k.*

PROOF: Direct application of MP: you have n choices for each object.

Example 1.2. Let A and B be any two sets. One denotes by B^A the set of all possible functions from A to B, i.e.:

$$B^A = \{f \mid f : A \to B \text{ is a function}\}.$$

Now suppose $|A| = m$ and $|B| = n$. Then $|B^A| = n^m$ (which also explains the notation). For determining a function from A to B involves choosing an element $f(a)$ of B for each element $a \in A$.

Proposition 1.3. *The number of ways to choose k objects from a set of n distinct objects, where repitition is not allowed and order of choice matters is*

$$P(n,\, k) = n(n-1)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i).$$

In particular, if $k > n$ then $P(n,\, k) = 0$ while if $k \leq n$ then $P(n,\, k) = \frac{n!}{(n-k)!}$. For $k = n$ we have $P(n,\, n) = n!$.

PROOF: Also a direct application of MP. Since repitition is not allowed the number of choices for an object reduces by one at each step.

Example 1.4. Recall that a function $f : A \to B$ is said to be *injective* if $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$ are distinct elements of A. Reasoning as in Example 1.1, we see that the number of injective functions from a finite set A to a finite set B is given by $P(|B|,\, |A|)$.

In particular, if $A = B$ and $|A| = n$ then there are $n!$ injective functions from A to itself. A function between finite sets of the same size is injective if and only if it is also *surjective*, hence *bijective*. A bijection from a set A (finite or infinite) to itself is called a *permutation* of A. Hence there are $n!$ permutations of an n-element set. This explains the notation "P".

Proposition 1.5. *The number of ways to choose k objects from a set of n distinct objects, where repitition is not allowed and order of choice doesn't matter is*

$$C(n,\, k) = \frac{P(n,\, k)}{k!} = \frac{n \cdot (n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}. \tag{1.1}$$

In particular, if $k \leq n$ then $C(n,\, k) = \frac{n!}{k!\,(n-k)!}$.

PROOF: Given an unordered choice of k distinct objects, there are then $k!$ ways to permute them to yield an ordered choice. This proves the first equality in (1.1). The rest is obvious.

Notation 1.6. "C" comes from "combination". It's more common to denote $C(n,\, k)$ by $\binom{n}{k}$ and refer to these numbers as *binomial coefficients*. We'll see why below.

Example 1.7. $\binom{n}{k}$ is simply the number of k-element subsets of an n-element set.

Many counting problems involve use of both AP and MP. To understand the next example you need to know that a "stryktipsrad" involves a guess of the outcomes of each of 13 football matches. There are 3 possible outcomes for each match: 1, X or 2.

Example 1.8. Let us compute the number of ways to fill in a stryktipsrad such that one gets at least 10 results right. One imagines the results of all 13 games being known to some superior being who can see into the future. Let A_{10}, A_{11}, A_{12}, A_{13} denote, respectively, the number of ways to fill in the tipskupong so that one gets 10, 11, 12, 13 correct. Then we are interested in the size of the union of these sets. Note that the sets are pairwise disjoint (by definition) so, by AP,

$$|A_{10} \cup A_{11} \cup A_{12} \cup A_{13}| = |A_{10}| + |A_{11}| + |A_{12}| + |A_{13}|. \tag{1.2}$$

Let us compute the size of A_{10} - the others are handled similarly. There are 13 results and 10 of these are correctly guessed. There are $\binom{13}{3}$ choices of the 3 incorrect results *and*, for each of these, there are 2 ways to guess incorrectly. Now MP applies, so

$$|A_{10}| = \binom{13}{3} \times 2^3.$$

Similarly,

$$|A_{11}| = \binom{13}{2} \times 2^2, \quad |A_{12}| = \binom{13}{1} \times 2^1, \quad |A_{13}| = \binom{13}{0} \times 2^0.$$

Substituting into (1.2), we find that

$$\begin{aligned} &|A_{10} \cup A_{11} \cup A_{12} \cup A_{13}| = \\ &= \binom{13}{3} \times 2^3 + \binom{13}{2} \times 2^2 + \binom{13}{1} \times 2^1 + \binom{13}{0} \times 2^0 = \\ &= \left(\frac{13 \times 12 \times 11}{1 \times 2 \times 3}\right) \times 8 + \left(\frac{13 \times 12}{1 \times 2}\right) \times 4 + 13 \times 2 + 1 \times 1 = \\ &= 2288 + 312 + 26 + 1 = 2627. \end{aligned}$$

So there are 2627 different ways to get at least 10 results right. Note that the total number of ways to fill in the tipskupong is 3^{13}, by Proposition 1.1, since there are 3 ways to fill in each row. Hence, the *probability* of getting at least 10 results right (if you just guess totally at random) is $\frac{2627}{3^{13}} \approx \frac{1}{607}$. We remark that the betting companies usually pay out to everyone who gets at least 10 result correct. However, the payout is usually *much* less than "607 gånger pengarna", which reflects two salient facts: (i) they want to make a profit (ii) the punters are usually people with some knowledge of football, they are not guessing randomly.

Proposition 1.9. *The number of ways to choose k objects from a set of n distinct objects, where repitition is allowed and order of choice doesn't matter is* $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

PROOF: Since order of choice is irrelevant but repitition is allowed, what matters is *how many* times each of the n distinct objects are chosen. To see where the formula comes from, we can observe that there is a 1-1 correspondence between the possible choices and sequences of $n + k - 1$ symbols of which k are identical "dots" and $n - 1$ are identical "dashes". For we can interpret the dashes as marking out where one "jumps" from one of the n distinct objects to the next. It's probably clearest with an example. Consider the following sequence of dots and dashes:

$$\bullet \bullet \mid \bullet \bullet \bullet \mid \mid \bullet \mid \bullet \bullet \bullet \bullet \mid \bullet .$$

This corresponds to a choice of 11 objects from a set of 6 distinct objects, where the number of times each object $1 - 6$ is chosen is, respectively, $2,\ 3,\ 0,\ 1,\ 4,\ 1$.

To complete the proof, note that the number of sequences of k dots and $n - 1$ dashes is obviously $\binom{n+k-1}{k}$ since one just needs to choose in which k positions to place the dots.

Balls and bins. Computer scientists *love* to talk in terms of placing balss in bins. We want to count the number of ways one can distribute k balls in n bins. There are four different cases, depending on whether or not the balls are *distinuighable*, and similarly for the bins. The problems are simplest when the bins are distinguishable, indeed in these cases we just have reformulations of problems already considered above. The cases when the bins are indistinguishable will be returned to later - see Remark 1.15 below.

Proposition 1.10. *The number of ways to distribute k distinguishable balls among n distinguishable bins is n^k.*

PROOF: This is just a reformulation of Proposition 1.1, since there are n choices for the bin in which to place each ball.

Proposition 1.11. *The number of ways to distribute k indistinguishable balls among n distinguishable bins is $\binom{n+k-1}{k}$.*

PROOF: This is just a reformulation of Proposition 1.9, since the indistinguishability of the balls means that all that matters is how many times one chooses each bin for a ball.

Example 1.12. Proposition 1.11 is used in statistical physics, where the balls are (elementary) particles and the bins are quantum energy levels. If you're interested, see `https://en.wikipedia.org/wiki/Bose-Einstein_statistics`

Example 1.13. Another interpretation of Proposition 1.11 is that $\binom{n+k-1}{k}$ is the number of solutions in non-negative integers to the equation

$$x_1 + x_2 + \cdots + x_n = k, \quad x_i \in \mathbb{N}_0. \tag{1.3}$$

For we can interpret x_i as the number of balls placed in bin number i. Solutions to (1.3) are usually referred to as *compositions of k into at most n parts*. The "at most" comes from the fact that the x_i are allowed to equal zero.

Example 1.14. Let $\mathbb{Z}^n$ denote the n-dimensional integer lattice, that is, the lattice of points in n-dimensional Euclidean space $\mathbb{R}^n$ all of whose coordinates are integers. We can interpret $\binom{n+k-1}{k}$ as the number of possible destinations of a k-step path in this lattice, starting from the origin and such that every step is in the positive direction along a coordinate axis. For we can interpret x_i in (1.3) as the number of steps taken along the i:th coordinate direction and note that, what determines where one ends up is simply the number of steps taken in each direction, not the order.

The technical terminology here would be to speak of the number of possible destinations for a k-step *simple, positively oriented, n-dimensional random walk*.

Remark 1.15. When counting the number of ways to place distinguishable balls into indistinguishable (a.k.a. identical) bins one encounters so-called *Stirling numbers*. When placing identical balls into identical bins one encounters so-called *integer partitions of k (into at most n parts)*. These problems are more difficult, no exact formulas are known but one can write down some recursive formulas. We will return to these topics in Lecture xx.