5th Lecture: 25/1

The binomial theorem for negative integer exponents. Our goal in the next one-and-a-half lectures is to show how the solution of linear recurrences can be carried out in a more systematic manner by means of so-called *generating functions*, and then to illustrate the greater power of such methods by applying them even to non-linear recurrences. A necessary starting point for this discussion is to extend the binomial theorem to situations where the exponent is not a positive integer. We are primarily interested in the case where the exponent is a negative integer, though even a rational exponent will make an appearance when we discuss the *Catalan numbers*.

Let us begin by observing that Proposition 2.1 can be rephrased as follows: Let $t \in \mathbb{N}$ and $x \in \mathbb{R}$. Then

$$(1+x)^t = \sum_{k=0}^{t} \binom{t}{k} x^k, \tag{5.1}$$

where

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}. \tag{5.2}$$

We make two observations:

Firstly, if $k > t$ then $\binom{t}{k} = 0$ obviously, since it is impossible to choose more than t objects. This agrees with the right-hand side of (5.2), which will be zero for $k > t$ since then $t = k$ will give a factor zero in the numerator. Hence, it is fine to write (5.1) as

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k. \tag{5.3}$$

Secondly, we observe that, for any fixed k, the RHS of (5.2) at the very least makes sense for any *real* number t, since it is just a polynomial in t of degree k. Hence both sides of (5.3) are at the very least well-defined functions of a real variable x, for any $t \in \mathbb{R}$, as long as x is in the domain of both. This is the case when $|x| < 1$ since then it's easy to see that the RHS of (5.3) converges. And, in fact, (5.3) does indeed hold for any $t \in \mathbb{R}$ and any $x \in (-1, 1)$, since it is just Taylor's theorem applied to the function $f(x) = (1+x)^t$.

In particular, let t be a negative integer, say $t = -n$, $n \in \mathbb{N}$. Replace x by $-x$ in (5.3). We get

$$|x| < 1 \;\Rightarrow\; \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k x^k, \tag{5.4}$$

where

$$\binom{-n}{k} \stackrel{\text{def}}{=} \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k \frac{(n+k-1)\cdots(n+1)n}{k!} = (-1)^k \binom{n+k-1}{k}.$$

Substituting this into (5.4) gives us what we want:

Theorem 5.1. *Let* $n \in \mathbb{N}$ *and* $x \in (-1,\, 1)$. *Then*

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k. \tag{5.5}$$

Though, as we said above, this follows from Taylor's theorem, it is really a *combinatorial* result, as the following proof shows.

COMBINATORIAL PROOF OF THEOREM 5.1: By the formula for a geometric series, if $|x| < 1$, then

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k. \tag{5.6}$$

Raise both sides to the n:th power:

$$(1-x)^{-n} = \left(\frac{1}{1-x}\right)^n = \left(\sum_{k=0}^{\infty} x^k\right)^n = \cdots = \sum_{k=0}^{\infty} c_k x^k,$$

where c_k is the number of ways one can get a term of x^k when one multiplies out the product of n factors[1]. To get x^k one must take x^{k_i} from the i:th factor, $i = 1,\, 2,\, \ldots,\, n$, such that $k_1 + k_2 + \cdots + k_n = k$. Hence, c_k is just the number of solutions to the equation

$$k_1 + k_2 + \cdots + k_n = k, \quad k_i \in \mathbb{N}.$$

But from Example 1.13 we know that the number of such solutions is $\binom{k+n-1}{k}$, v.s.v.

Remark 5.2. Yet another way to prove Theorem 5.1 is to differentiate both sides of (5.6) $n-1$ times. The LHS will become $(n-1)!(1-x)^{-n}$. The RHS can be differentiated termwise (because of uniform convergence). Each of the terms $1,\, x,\, \ldots,\, x^{n-2}$ will collapse to zero after $n-1$ differentiations. For any $k \geq n-1$, $\frac{d^{n-1}}{dx^{n-1}}(x^k) = k(k-1)\ldots(k-n+2)x^{k-(n-1)}$. Thus

$$(n-1)! \cdot (1-x)^{-n} = \sum_{k=n-1}^{\infty} k(k-1)\ldots(k-n+2)\, x^{k-(n-1)}.$$

Divide both sides by $(n-1)!$ and change the summation index from k to $l := k-(n-1)$. We get

$$(1-x)^{-n} = \sum_{l=0}^{\infty} \frac{(l+n-1)(l+n-2)\ldots(l+1)}{(n-1)!}\, x^l.$$

The coefficient of x^l is just $\binom{l+n-1}{n-1}$, which is the same as $\binom{l+n-1}{l}$.

Remark 5.3. The statement that, for any $t \in \mathbb{R}$ and $|x| < 1$,

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k, \quad \text{where } \binom{t}{k} \text{ is as defined in (5.2),} \tag{5.7}$$

[1] Strictly speaking, one uses here the uniform convergence of the geometric series in any closed subinterval of $(-1,\, 1)$, since then Weierstrass theorem implies we can rearrange terms as we wish when we multiply out.

is referred to as the *Generalized Binomial Theorem.* In this course we will mostly only use the version (5.5), which is more convenient when the exponent is a negative integer. However, in at least one place (when we discuss the Catalan numbers), we will employ version (5.7) with $t = 1/2$.

Generating Functions. We begin with the general definition.

Definition 5.4. Let $(u_n)_{n=1}^{\infty}$ be a sequence of complex numbers. The *generating function* $G(x)$ for the sequence is given by the power series $G(x) = \sum_{n=0}^{\infty} u_n x^n$.

One can, of course, complain that this is not a precise definition, since the domain of the function $G(x)$ has not been specified - indeed, the domain will depend on the particular sequence. This is true, but:

(i) In all our examples, the sequence (u_n) will grow at most exponentially. If $u_n = O(c^n)$ then $G(x)$ is absolutely convergent for $|x| < 1/c$.

(ii) One can get around this issue completely, yet rigorously, by considering $G(x)$ as what is called a *formal power series*, i.e.: an element in a *formal power series ring* $\mathbb{C}[[x]]$. I don't want to go into what this means for the moment, but might return to it later.

Example 5.5. Consider the recurrence

$$u_0 = 1, \quad u_1 = 3, \quad u_n = 6u_{n-1} - 8u_{n-2} \ \ \forall\, n \geq 2.$$

This is the simplest possible kind of linear recurrence: homogeneous, with no repeated roots. The auxiliary equation is $x^2 - 6x + 8 = 0$, with roots $x_1 = 2$, $x_2 = 4$. Hence the solution is of the form

$$u_n = C_1 \cdot 2^n + C_2 \cdot 4^n.$$

Inserting the initial conditions gives

$$\begin{aligned} n = 0: \quad u_0 = 1 = C_1 + C_2, \\ n = 1: \quad u_1 = 3 = 2C_1 + 4C_2, \end{aligned}$$

and hence $C_1 = C_2 = 1/2$. Thus,

$$u_n = \frac{1}{2}(2^n + 4^n). \tag{5.8}$$

We now illustrate how to get the same result using the generating function method. Though it may seem less efficient, the point is that it should also seem less “ad hoc”, and should give greater insight into why the solution has the form it does (no ”guessing” is involved). So let $G(x) := \sum_{n=0}^{\infty} u_n x^n$.

Step 1: Use the recursion to express $G(x)$ as a rational function, i.e.: a quotient of polynomials. Given the initial conditions we can write:

$$G(x) = u_0 + u_1 x + \sum_{n=2}^{\infty} u_n x^n = 1 + 3x + \sum_{n=2}^{\infty} u_n x^n \quad (5.9)$$

$$xG(x) = \sum_{n=0}^{\infty} u_n x^{n+1} = \sum_{n=1}^{\infty} u_{n-1} x^n = u_0 x + \sum_{n=2}^{\infty} u_{n-1} x^n = x + \sum_{n=2}^{\infty} u_{n-1} x^n \quad (5.10)$$

$$x^2 G(x) = \sum_{n=0}^{\infty} u_n x^{n+2} = \sum_{n=2}^{\infty} u_{n-2} x^n. \quad (5.11)$$

From (5.9), (5.10) and (5.11) it follows that

$$(1 - 6x + 8x^2)G(x) = (1 + 3x) - 6x + \sum_{n=2}^{\infty} (u_n - 6u_{n-1} + 8u_{n-2}) x^n.$$

The recursion implies that every coefficient in the sum is zero. Hence

$$(1 - 6x + 8x^2)G(x) = 1 - 3x \;\Rightarrow\; G(x) = \frac{1 - 3x}{1 - 6x + 8x^2} = \frac{1 - 3x}{(1 - 2x)(1 - 4x)}.$$

Step 2: Make a *partial fraction decomposition* of the rational function. In the case at hand, this means finding constants A and B such that

$$\frac{1 - 3x}{(1 - 2x)(1 - 4x)} = \frac{A}{1 - 2x} + \frac{B}{1 - 4x}.$$

Multiplying through by $(1 - 2x)(1 - 4x)$ we have the requirement that

$$\begin{aligned} 1 - 3x &= A(1 - 4x) + B(1 - 2x) \\ \Rightarrow 1 - 3x &= (A + B) + x(-4A - 2B) \\ \Rightarrow A + B = 1 \quad &\text{and} \quad 4A + 2B = 3 \\ &\Rightarrow A = B = 1/2. \end{aligned}$$

Thus,

$$G(x) = \frac{1}{2}\left[\frac{1}{1 - 2x} + \frac{1}{1 - 4x}\right] = \frac{1}{2}[(1 - 2x)^{-1} + (1 - 4x)^{-1}].$$

Step 3: Use Theorem 5.1 to rewrite this as a power series. We have

$$G(x) = \frac{1}{2}\left[\sum_{k=0}^{\infty} (2x)^k + \sum_{k=0}^{\infty} (4x)^k\right].$$

By comparing the coefficients of x^n, which by definition is equal to u_n on the LHS, we get, as in (5.8),

$$u_n = \frac{1}{2}(2^n + 4^n), \quad \text{v.s.v.}$$

Example 5.6. Things get more complicated when either (i) the auxiliary equation has repeated roots or (ii) the recurrence is inhomogeneous. However, it turns out one can always express $G(x)$ as a rational function as long as the inhomogeneous part consists of only exponential functions and polynomials. This explains “why” ILRs can be solved in just these cases. To see how things work, see `Demo2.pdf` for a re-solution of Example 4.5 using generating functions.

Catalan numbers. So far we have seen how the introduction of generating functions allows us to approach and to understand the solution of linear recurrence relations in a more methodical manner (sometimes at the cost of longer calculations if done by hand, but easily programmable in a computer). Another advantage of generating functions is that they can sometimes shed light even on more complicated looking recurrences. The Catalan numbers provide a classical illustration of this. We first define them via a sequence of three definitions:

Definition 5.7. A *diagonal path* in the two-dimensional integer lattice $\mathbb{Z}^2$ is a path for which each step is of one of the following four types:

$$\begin{gathered}(x,\,y)\to(x+1,\,y+1)\quad\text{(up and to the right)},\\(x,\,y)\to(x+1,\,y-1)\quad\text{(down and to the right)},\\(x,\,y)\to(x-1,\,y+1)\quad\text{(up and to the left)},\\(x,\,y)\to(x-1,\,y-1)\quad\text{(down and to the left)}.\end{gathered}$$

Definition 5.8. A diagonal path in $\mathbb{Z}^2$ is called a *Dyck path* if it satisfies the following two requirements:
(i) every step is to the right
(ii) the path never goes below the x-axis, though it is allowed to touch the x-axis.

Definition 5.9. Let $n \in \mathbb{N}$. The *Catalan number* C_n is defined as the number of Dyck paths from $(0,\,0)$ to $(2n,\,0)$.

One can compute the first few Catalan numbers by hand[2]: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$ etc - see Figure 5.1. It is probably not at all obvious at this stage that there is a beautiful general formula for this sequence, namely

Theorem 5.10.

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{5.12}$$

We will give two beautiful proofs of this result next day. The first, "algebraic", proof relies on the following recurrence relation for the C_n:

Proposition 5.11.

$$\forall\, n \geq 1: \quad C_n = \sum_{m=1}^{n} C_{m-1}C_{n-m}. \tag{5.13}$$

PROOF: Let $n \geq 1$ and consider a Dyck path from $(0,\,0)$ to $(2n,\,0)$. Let $(2m,\,0)$ be the first point at which the path returns to the x-axis after leaving the origin: thus $1 \leq m \leq n$.

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[2] If you go to `https://oeis.org` and just write in $1,\,1,\,2,\,5$ the database will already recognise the sequence and return with a wealth of information about the Catalan numbers.

The part of the path from $(2m, 0)$ to $(2n, 0)$ can be considered a Dyck path of length $2(n-m)$. Hence there are C_{n-m} possibilities for this part of the path.

Consider the part of the path between $(0, 0)$ and $(2m, 0)$. The first step must be up to $(1, 1)$ and the last step down from $(2m, 1)$. By assumption, the path does not touch the x-axis in between $x = 1$ and $x = 2m-1$. Hence the part of the path between $(1, 1)$ and $(2m, 1)$ can be considered a Dyck path of length $2(m-1)$, shifted upward one unit. So there are C_{m-1} possibilities for this part of the path.

Hence, by MP, there are $C_{m-1}C_{n-m}$ possibilities for a Dyck path of length $2n$ which first returns to the x-axis after $2m$ steps. Together with AP, this proves (5.13). The argument is summarized in Figure 5.2.

Remark 5.12. There are a total of $\binom{2n}{n}$ rightward diagonal paths from $(0, 0)$ to $(2n, 0)$, since any such path contains exactly n upward and n downward steps. Hence, (5.13) says that a fraction $\frac{1}{n+1}$ of these paths have the property that they never go under the x-axis.