4th Lecture: 25/1

In order to prove Theorem 3.8, we need a lemma.

Lemma 4.1. *Let $p(x) \in \mathbb{C}[x]$ be a polynomial and $\alpha \in \mathbb{C}$ a root of multiplicity $l > 0$. Then α is also a root, of multiplicity $l - i$, of the i:th derivative $p^{(i)}(x)$, for each $1 \leq i \leq l - 1$.*

PROOF OF LEMMA: That α is a root of multiplicity l means that $x - \alpha$ is a factor of multiplicity l, in other words that $p(x) = (x - \alpha)^l q(x)$, for some polynomial $q(x)$. It is then easy to see that, if we differentiate i times (use the product rule and induction on i), $(x - \alpha)^{l-i}$ will still be a factor of $p^{(i)}(x)$, and hence α still be a root of $p^{(i)}(x)$ of multiplicity $l - i$, as long as $i < l$.

PROOF OF THEOREM 3.8: Set $p_0(x) := x^k - \sum_{i=1}^{k} c_i x^{k-i}$ and let α be a root of $p_0(x)$ of some multiplicity $l > 0$. Since (3.7) is a linear recurrence, it suffices to prove that $a_n = n^i \cdot \alpha^n$ solves (3.7), for each $0 \leq i \leq l - 1$. This is immediately clear for $i = 0$, by what we already discussed in Lecture 3, namely: direct insertion into (3.7) and cancellation of α^{n-k} yields α as a solution of (3.8).

Now suppose $l > 1$ and define a sequence of polynomials $P_0(x),\ P_1(x),\ \ldots,\ P_{l-1}(x)$ recursively as follows:

$$P_0(x) := x^{n-k} p_0(x),$$
$$P_i(x) := x P'_{i-1}(x), \quad \text{for } i = 1,\ \ldots,\ l-1.$$

Since $c_k \neq 0$ (as otherwise the degree of the recurrence would be less than k), it follows that $\alpha \neq 0$. Hence α is also a root of multiplicity l of $P_0(x)$. Hence, by Lemma 4.1, α is a root of multiplicity $l - 1$ of $P'_0(x)$ and hence of $P_1(x)$. But, computing directly, we have

$$P_1(x) = x \cdot \frac{d}{dx}\left(x^n - \sum_{i=1}^{k} c_i x^{n-i}\right) = n \cdot x^n - \sum_{i=1}^{k} c_i (n-i) x^{n-i}$$

and hence (since $l > 1$),

$$0 = P_1(\alpha) = n \cdot \alpha^n - \sum_{i=1}^{k} c_i (n-i) \alpha^{n-i}.$$

This just says that $a_n = n \cdot \alpha^n$ satisfies (3.7), as desired.

One can now continue in the same manner and check that, for each $1 \leq i \leq l - 1$, α is a root of $P_i(x)$ of multiplicity $l - i$, that

$$P_i(x) = n^i x^n - \sum_{j=1}^{k} c_j (n-j)^i x^{n-j}$$

and hence, since $0 = P_i(\alpha)$, that $a_n = n^i \cdot \alpha^n$ satisfies (3.7), v.s.v.

Inhomogeneous linear recurrences (ILRs). We are now concerned with recursions of the form

$$a_{n+k} = \sum_{i=1}^{k} c_i a_{n+k-i} + b_n \tag{4.1}$$

where $c_1, c_2, \ldots, c_k$ are constants as before, and $(b_n)_{n=0}^{\infty}$ is, a priori, any sequence of complex numbers. The homogeneous case, studied in Lecture 3, is the case $b_n = 0$ for all n. We saw that in that case the general solution to (4.1) is a k-dimensional vector subspace V of l^{∞}. More generally, we have

Proposition 4.2. *The general solution to (4.1) is an element of*

$$V + \boldsymbol{x}_p = \{\boldsymbol{v} + \boldsymbol{x}_p : \boldsymbol{v} \in V\},$$

where V is the general solution to the homogeneous part ($b_n := 0$ for all n) and $\boldsymbol{x}_p \in l^{\infty}$ is any solution whatsoever to (4.1).

The proof is simple linear algebra and left to the reader. Note that $\boldsymbol{x}_p$ is called a *particular solution* to (4.1) and the full solution set $V + \boldsymbol{x}_p$ is referred to as a *coset* of the subspace V of l^{∞}.

The Proposition reduces the solution of an ILR to that of a corresponding HLR, plus the finding of *any* particular solution. Usually it is not possible to give an explicit formula for a particular solution, but there are two cases in which it is so:

Theorem 4.3. *(i) Suppose $b_n = \alpha^n$ for some fixed $\alpha \in \mathbb{C}$. If α is a root of multiplicity $l \geq 0$ of the homogeneous part of (4.1), then there is some constant C such that $b_n = C \cdot n^l \cdot \alpha^n$ solves (4.1).*
(ii) Suppose $b_n = p(n)$, for some fixed polynomial $p(x) \in \mathbb{C}[x]$. Let $l \geq 0$ be the multiplicity of 1 as a root of the characteristic equation of the homogeneous part of (4.1). Then there is some polynomial $q(x)$, of the same degree as $p(x)$, such that $b_n = n^l \cdot q(n)$ solves (4.1).

Theorem 4.3 can be proven in a similar manner to Theorem 3.8, but we choose to omit the details.

Example 4.4. In this example we first derive an ILR (combinatorial problem) before solving it (algebraic problem).

For every $n \geq 0$, let q_n be the number of n-letter words in the alphabet $\mathcal{A} = \{a, b, c, d\}$ which have an odd number of b's. We can compute the first few values directly:

$q_0 = 0$: The empty word has an even numbers (zero) of b's.
$q_1 = 1$: The word must be b.
$q_2 = 6$: One letter must be a b and the other something else (3 choices). Order matters (2 choices). MP $\Rightarrow 3 \cdot 2 = 6$. The allowed words are ab, cb, db, ba, bc, bd.

I claim that, for every $n \geq 1$,

$$q_n = 2 \cdot q_{n-1} + 4^{n-1}. \tag{4.2}$$

For an admissable word of length n, consider the following two cases:

Case 1: The word begins with a b. Then the remaining letters form a word of length $n-1$ with an even number of b's. There are 4^{n-1} words of length $n-1$ in total (by MP) and q_{n-1} of them, by definition, have an odd number of b's. Hence, the number of admissable words in Case 1 is $4^{n-1} - q_{n-1}$.

Case 2: The word begins with a, c or d. Then, by a similar analysis to Case 1, there are q_{n-1} possibilities for the remaining letters. Since there are three choices for the first letter, by MP there are a total of $3 \cdot q_{n-1}$ admissable words in Case 2.

Clearly, Cases 1 and 2 are mutually exclusive and exhaust all options so, by AP, the total number of admissable words of length n is $(4^{n-1} - q_{n-1}) + 3 \cdot q_{n-1} = 2 \cdot q_{n-1} + 4^{n-1}$, which proves (4.2).

So now we are left to solve the ILR

$$q_0 = 0, \quad q_n = 2q_{n-1} + 4^{n-1} \ \ \forall\, n \geq 1. \tag{4.3}$$

Step 1: Find the general solution of the homogeneous part of (4.3), namely of $q_n = 2q_{n-1}$. The characteristic equation is $x = 2$, so the general solution is

$$q_{h,n} = C_1 \cdot 2^n. \tag{4.4}$$

Step 2: Find a particular solution of (4.3). We apply Theorem 4.3(i). Here $\alpha = 4$, which is not a root (i.e.: a root of multiplicity zero) of the characteristic equation. Hence the particular solution has the form $q_{p,n} = C \cdot 4^n$. To find C we substitute into (4.3) and get

$$C \cdot 4^n = 2C \cdot 4^{n-1} + 4^{n-1} \ \Rightarrow \ C = -1/2.$$

Step 3: By Proposition 4.2, the general solution to (4.3) is

$$q_n = q_{h,n} + q_{p,n} = C_1 \cdot 2^n - \frac{1}{2} \cdot 4^n.$$

To find C_1 we insert the initial condition:

$$q_0 = 0 = C_1 - \frac{1}{2} \ \Rightarrow \ C_1 = \frac{1}{2}.$$

Hence, $q_n = \frac{1}{2}(4^n - 2^n)$. Note that q_n is "slightly less than half of" 4^n, and that the latter is the total number of words of length n in a 4-letter alphabet. Is this what you would expect intuitively ?

Example 4.5 (see Demo1). Solve the recurrence

$$u_0 = u_1 = 1, \quad u_n = 6u_{n-1} - 5u_{n-2} + 5^n + n + 1 \ \ \forall\, n \geq 2. \tag{4.5}$$

Step 1: The characteristic equation is $x^2 = 6x - 5$, with roots $\alpha_1 = 1$, $\alpha_2 = 5$. Hence

$$u_{h,n} = C_1 \cdot 1^n + C_2 \cdot 5^n = C_1 + C_2 \cdot 5^n.$$

Step 2: By linearity, one can take $u_{p,n} = u_{p_1,n} + u_{p_2,n}$, where $u_{p_1,n}$ is a particular solution corresponding to $b_n = 5^n$ and $u_{p_2,n}$ is a particular solution corresponding to $b_n = n + 1$.

By Theorem 4.3(i), since $\alpha = 5$ is a root of multiplicity one of the characteristic equation, we should take $u_{p_1,n} = C_3 \cdot n \cdot 5^n$. Substitute:

$$u_n = 6u_{n-1} - 5u_{n-2} + 5^n \;\Rightarrow\; C_3 \cdot n \cdot 5^n = 6C_3 \cdot (n-1) \cdot 5^{n-1} - 5C_3 \cdot (n-2) \cdot 5^{n-2} + 5^n.$$

If one multiplies out one will find that coefficients of $n \cdot 5^n$ cancel exactly. Then comparing coefficients of 5^n yields

$$0 = -\frac{6}{5}C_3 + \frac{10}{25}C_3 + 1 \;\Rightarrow\; C_3 = \frac{5}{4}.$$

By Theorem 4.3(ii), since 1 is a root of multiplicity one of the characteristic equation, one should take $u_{p_2,n} = n \cdot (C_4 n + C_5) = C_4 n^2 + C_5 n$. Substitute:

$$u_n = 6u_{n-1} - 5u_{n-2} + (n+1) \;\Rightarrow$$
$$C_4 n^2 + C_5 n = 6[C_4(n-1)^2 + C_5(n-1)] - 5[C_4(n-2)^2 + C_5(n-2)] + n + 1.$$

Multiply out and compare coefficients of the different powers of n. One will find that the coefficients of the highest power n^2 cancel exactly. In comparing coefficents of n one will find that C_5 cancels exactly and be left with $C_4 = -1/8$. Finally, comparing constant coefficients yields

$$0 = (6C_4 - 6C_5) + (-20C_4 + 10C_5) + 1 \;\overset{C_4=-1/8}{\Rightarrow}\; C_5 = -11/16.$$

Step 3: Hence, by Proposition 4.2, the general solution to our recurrence is

$$u_n = u_{h,n} + u_{p,n} = C_1 + C_2 \cdot 5^n + \frac{5}{4} \cdot n \cdot 5^n - \frac{n^2}{8} - \frac{11n}{16}.$$

We find C_1, C_2 by inserting the initial conditions:

$$n = 0: \quad u_0 = 1 = C_1 + C_2,$$
$$n = 1: \quad u_1 = 1 = C_1 + 5C_2 + \frac{25}{4} - \frac{1}{8} - \frac{11}{16} \;\Rightarrow\; C_1 + 5C_2 = -\frac{71}{16}.$$

Solving, one gets $C_1 = 151/64$, $C_2 = -87/64$. Hence,

$$u_n = \left(-\frac{87}{64} + \frac{5n}{4}\right) \cdot 5^n - \frac{n^2}{8} - \frac{11n}{16} + \frac{151}{64}.$$