First Exercise Session: 27/3

Theme: Combinatorics (inkl. Inclusion-Exclusion)

Relevant Chapters: Vol. 1: 5

1. Peter often visits the basement in Akademibokhandeln in Nordstan and then usually finds himself standing before one of the following five shelves: Naturvetenskap (N), Samhälle (S), Politik (P), Ekonomi (E) or Historia (H).

Assume that each of these shelves contains 100 different books (in parts **(a)**-**(d)** below, we assume there is only one copy of each book), that every book on the N and S shelves costs 100kr and that every book on the P, E and H shelves costs 50kr.

(a) In how many ways can Peter buy 6 books and line them up on his own shelf at home from left to right ?
(b) In how many ways can Peter buy 6 books if all we care about is how many he chooses from each of the 5 categories and not the exact titles ? (This time he just throws them in a pile at home).
(c) In how many ways can Peter spend exactly 200kr ?
(d) If Peter picks 6 books at random, what is the probability he picks the same number of books from the N and S shelves ?
(e) How many different collections of 6 books could Peter purchase if instead there were 5 copies of every book on the shelves (hence 20 different titles on each shelf).
(f) Same question as in **(e)** but instead 12 books are purchased.

2. In Pythonland there are 10 political parties, including the Judaean People's Front (JPF) and the People's Front of Judaea (PFJ). Parliament has 15 seats which are assigned geographically, one to each of 15 constituencies.

(a) If the seats were to be distributed completely at random amongst the 10 parties, in how many ways can this be done if if matters which constituency is represented by which party ?
(b) Same question as in **(a)** but dropping the last assumption.
(c) Under the same assumptions as in **(a)**, what is the probability that both JPF and PFJ get exactly 5 seats and that no other party gets more than one seat ?
(d) In the last election, JPF got 4 seats, PFJ got 3 seats and every other party got one seat. In how many ways can a coalition be built which together have 8 seats, assuming PFJ and JPF are bitter enemies who under no circumstances will go into government together ?

3. How many words (i.e.: strings of letters, don't need to mean anything) can be made from DISCRETEMATH in which none of the words IS, CRETE, AT appears as a substring of consecutive letters ?

Solutions

1. (a) Order of choice matters, replacement not allowed: $P(500,\,6)$.
(b) Order of choice doesn't matter (not lined up) and replacement (of categories) allowed. So we count solutions to $\sum_{i=1}^{5} x_i = 6$, where x_i is the number of books bought in category i. ANSWER: $\binom{6+5-1}{5-1} = \binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 210$.
(c) We divide into three cases:

CASE 1: Buys 2 books worth 100kr each. There are $\binom{200}{2}$ choices for these two.

CASE 2: Buys 1 book worth 100kr and 2 books worth 50kr. There are 200 choices for the first book, then $\binom{300}{2}$ choices for the other two.

CASE 3: Buys 4 books worth 50kr. There are $\binom{300}{4}$ choices.

By AP and MP, the total number of possible purchases is

$$\binom{200}{2} + 200 \cdot \binom{300}{2} + \binom{300}{4}.$$

(d) The probability is $A/\binom{500}{6}$, where A is the number of ways of picking the 6 books so that one picks an equal number from N and S. This in turn leaves four cases:

CASE 1: No N or S books. Then all 6 are chosen from the remaining 300, so $\binom{300}{6}$ possibilities.

CASE 2: One each of N and S. By MP, the number of possibilities is $100 \cdot 100 \cdot \binom{300}{4}$.

CASE 3: Two each of N and S. By MP there are $\binom{100}{2} \cdot \binom{100}{2} \cdot \binom{300}{2}$ possibilities.

CASE 4: Three each of N and S. By MP there are $\binom{100}{3}^2$ possibilities.

Finally, by AP we have that the desired probability is

$$\frac{\binom{300}{6} + 100^2 \cdot \binom{300}{4} + \binom{100}{2}^2 \cdot \binom{300}{2} + \binom{100}{3}^2}{\binom{500}{6}}.$$

(e) There are a total of 100 titles. Let x_i be the number of copies bought of title i. We seek the number of solutions to

$$\sum_{i=1}^{100} x_i = 6, \quad 0 \leq x_i \leq 5 \; \forall\, i.$$

Without the upper bound on the x_i, the number of solutions would be $\binom{100+6-1}{6} = \binom{105}{6}$. We subtract the number of solutions in which some $x_i > 5$. But it is clear there are exactly 100 such solutions, in each of which some $x_i = 6$ and all other $x_i = 0$. ANSWER: $\binom{105}{6} - 100$.
(f) This time we count the number of solutions to

$$\sum_{i=1}^{100} x_i = 12, \quad 0 \leq x_i \leq 5 \; \forall\, i. \tag{1}$$

Without the upper bound on the x_i there would be $\binom{100+12-1}{12} = \binom{111}{12}$ solutions. For each $i = 1,\,\ldots,\,100$, let

$$A_i = \{\text{solutions to (1) where } x_i \geq 6\}.$$

Then the answer to the question is

$$\binom{111}{12} - \left|\bigcup_{i=1}^{100} A_i\right|. \tag{2}$$

By the Inclusion-Exclusion principle,

$$\left|\bigcup_{i=1}^{100} A_i\right| = \sum_{i=1}^{100} |A_i| - \sum_{i\neq j} |A_i \cap A_j| + \dots. \tag{3}$$

Note that all remaining terms in the I-E expansion will be zero since $A_i \cap A_j \cap A_k$ consists of those solutions with each of x_i, x_j and x_k being at least 6, which is impossible since the x_i:s sum to only 12. Note that from this reasoning we also see immediately that

$$|A_i \cap A_j| = 1 \; \forall \, i \neq j. \tag{4}$$

Finally consider some A_i. Let $y_i := x_i - 6$. Then A_i consists of all solutions to

$$y_i + \sum_{j\neq i} x_j = 6, \quad y_i \geq 0, \, x_j \geq 0.$$

Hence,

$$|A_i| = \binom{100+6-1}{6} = \binom{105}{6} \; \forall \, i. \tag{5}$$

Inserting (3), (4) and (5) into (2), we find that the answer is

$$\binom{111}{12} - 100 \cdot \binom{105}{6} + \binom{100}{2}.$$

2. (a) 10^{15}.
(b) All that matters is the *number* of seats assigned to each party. Hence we are counting solutions to $\sum_{i=1}^{10} x_i = 15$, $x_i \geq 0$. ANSWER: $\binom{15+10-1}{10-1} = \binom{24}{9}$.
(c) The probability is $A/10^{15}$, where A is the number of seat assignments satisfying our conditions. To compute A, we can reason for example as follows:
- first assign 5 seats to JPF: $\binom{15}{5}$ possibilities
- next assign 5 of the remaining 10 seats to PFJ: $\binom{10}{5}$ possibilities
- finally, choose in order 5 of the remaining 8 parties to receive one seat each: $P(8, 5)$ possibilities.

Hence, by MP, the desired probability is

$$\frac{\binom{15}{5} \cdot \binom{10}{5} \cdot P(8, 5)}{10^{15}}.$$

(d) Since JPF and PFJ can't work together, there are three options for forming a government with exactly 8 seats:

CASE 1: JPF and 4 minor parties. There are $\binom{8}{4} = 70$ ways to choose the 4 minors.

CASE 2: PFJ and 5 minor parties. There are $\binom{8}{5} = \binom{8}{3} = 56$ ways to choose the 5 minors.

CASE 3: If neither JPF nor PFJ are involved, then all 8 minor parties must join up. Thus this leaves exactly 1 option.

By AP, the total number of possible coalitions is $70 + 56 + 1 = 127$.

3. Since there are two Es and two Ts, the total number of possible words, without any extra conditions, is $\frac{12!}{2!\,2!}$. Let A, B resp. C denote the set of words containing IS, CRETE resp. AT. Then the number of admissable words is $\frac{12!}{2!\,2!} - |A \cup B \cup C|$, which by the I-E principle equals

$$\frac{12!}{2!\,2!} - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|. \tag{6}$$

We take the seven unknown terms one-by-one.

A: We can consider IS as one letter. Then we're forming words with 11 instead of 12 letters. There are still two Es and two Ts. So the number of words is $\frac{11!}{2!\,2!}$.

B: We can consider CRETE as one letter. Then we're forming words with 8 instead of 12 letters. There are no longer any copies of an E or a T. So the number of words is $8!$.

C: We can consider AT as one letter. Then we're forming words with 11 instead of 12 letters. There are still two Es, but now just one T. So the number of words is $\frac{11!}{2!}$.

$A \cap B$: We can consider IS and CRETE as one letter each. Then we're forming words with 7 instead of 12 letters. There are no copies of E or T left. So the number of words is $7!$.

$A \cap C$: We can consider IS and AT as one letter each. Then we're forming words with 10 instead of 12 letters. There are still two Es. So the number of words is $\frac{10!}{2!}$.

$B \cap C$: We can consider CRETE and AT as one letter each. Then we're forming words with 7 instead of 12 letters. There are no copies of E or T left. So the number of words is $7!$.

$A \cap B \cap C$: We can consider IS, CRETE and AT as one letter each. Then we're forming words with 6 instead of 12 letters. There are no copies of E or T left. So the number of words is $6!$.

Inserting everything into (0.6), we get the answer

$$\frac{12!}{2!\,2!} - \left(\frac{11!}{2!\,2!} + 8! + \frac{11!}{2!}\right) + \left(7! + \frac{10!}{2!} + 7!\right) - 6!$$