**Second Exercise Session: 31/1**

**Theme: Pigeonhole Principle, Linear Recursions**

**Relevant Chapters: Vol. 1: 5.3, Vol.2: 4.2**

**1. (5.55 in Vol. 1)** Prove that if 17 or more pieces are placed on a chessboard, then there must be two which are adjacent, either vertically, horizontally or diagonally.

**2.** Assuming that "acquaintanceship" is a symmetric relation (I know you if and only if you know me), prove that in any group of 6 people, there must be subgroup of 3 who are either all mutual acquaintances or mutual strangers.

(HINT: Consider the situation from the viewpoint of one of the 6 people.)

**3.** For each $n \geq 0$, let $q_n$ be the number of $n$-letter words in the alphabet $\{a, b, c\}$ which don't contain any pair of consecutive $b$'s. Find and solve a recursion for the numbers $q_n$.

**4.** Consider the recursion

$$a_0 = a_1 = a_2 = 1, \quad a_n = 3a_{n-1} - 4a_{n-3} \, \forall \, n \geq 3.$$

**(i)** Compute $a_3$ and $a_4$ directly.
**(ii)** Determine an exact formula for $a_n$.

**5.** Consider the recursion

$$a_0 = 0, \; a_1 = 1, \quad a_n = 2a_{n-1} - a_{n-2} \, \forall \, n \geq 2.$$

**(i)** Determine the formula for $a_n$ "by staring".
**(ii)** Verify the formula by solving the recursion in the usual manner.

## Solutions

**1.** Divide the $8 \times 8$ chessboard into sixteen $2 \times 2$ squares. Since we place 17 pieces, by the Pigeonhole Principle, there must be some square in which we place at least 2 pieces. But these two pieces must then be adjacent.

**2.** Isolate one of the 6 people, call him P. There are 5 others and, since $5 > 2 \cdot 2$, by the Extended Pigeonhole Principle, one of the following must occur:

*Case 1:* P has at least 3 acquaintances.

*Case 2:* At least 3 others are strangers to P.

First consider Case 1. If any two amongst P's acquaintances are also acquainted, then these two together with P form a group of 3 mutual acquaintances and we're done. The only way to avoid this is if all of P's acquaintances are mutual strangers, but since there are at least 3 of them, we must then have a group of 3 mutual strangers.

Case 2 is handled completely analogously, we just interchange "acquaintance" and "stranger".

**3.** We have the initial conditions

$q_0 = 1$: the empty word works.

$q_1 = 3$: any of $a$, $b$, $c$ is okay.

To obtain a recursion, consider a word of length $n$ satisfying our condition and two cases:

*Case 1:* The last letter is $a$ or $c$.

*Case 2:* The last letter is $b$.

In Case 1, the first $n-1$ letters must satisfy the same condition as at the outset but no others. Hence there are $q_{n-1}$ possibilities for this part of the word. There are two possible letters in the last position hence, by MP, a total of $2q_{n-1}$ possible words.

In Case 2, the second to last letter cannot also be $b$, so it must be $a$ or $c$. The remaining $n-2$ letters must satisfy the same condition as at the outset, but no others. Hence there are $q_{n-2}$ possibilities for the first $n-2$ letters and two possibilities for the $(n-1)$:st letter, so $2q_{n-2}$ possible words in all.

Finally then, by AP, we have a total of $2q_{n-1} + 2q_{n-2}$ possible words of length $n$, which means that

$$q_n = 2q_{n-1} + 2q_{n-2}.$$

The characteristic equation is $\alpha^2 = 2\alpha + 2$, which has the roots $\alpha_{1,2} = 1 \pm \sqrt{3}$. Hence the general solution is

$$q_n = C_1 \cdot (1+\sqrt{3})^n + C_2 \cdot (1-\sqrt{3})^n.$$

Insert the initial conditions:

$$n = 0: \quad q_0 = 1 = C_1 + C_2,$$
$$n = 1: \quad q_1 = 3 = (1+\sqrt{3})C_1 + (1-\sqrt{3})C_2.$$

Two linear equations in two unknowns, which are easily solved to yield $C_1 = \frac{\sqrt{3}+2}{2\sqrt{3}}$, $C_2 = \frac{\sqrt{3}-2}{2\sqrt{3}}$. Hence,

$$q_n = \frac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n.$$

**4. (i)** According to the recursion and with the given initial conditions:

$$n = 3: \quad a_3 = 3a_2 - 4a_0 = 3(1) - 4(1) = -1,$$
$$n = 4: \quad a_4 = 3a_3 - 4a_1 = 3(-1) - 4(1) = -7.$$

**(ii)** STEP 1: The characteristic equation is $\alpha^3 = 3\alpha^2 - 4$. One can perhaps see quickly that $\alpha_1 = -1$ is a root, so $\alpha + 1$ is a factor of the polynomial. Standard polynomial division gives

$$\frac{\alpha^3 - 3\alpha^2 + 4}{\alpha + 1} = \alpha^2 - 4\alpha + 4 = (\alpha - 2)^2$$

so $\alpha_{2,3} = 2$ is a repeated root. Hence, the general solution to the recursion is

$$a_n = C_1 \cdot (-1)^n + C_2 \cdot 2^n + C_3 \cdot n \cdot 2^n.$$

STEP 2: We insert the initial conditions:

$$n = 0: \quad a_0 = 1 = C_1 + C_2,$$
$$n = 1: \quad a_1 = 1 = -C_1 + 2C_2 + 2C_3,$$
$$n = 2: \quad a_2 = 1 = C_1 + 4C_2 + 8C_3.$$

Three linear equations in three unknowns, so apply standard Gauss elimination to get $C_1 = -1/9$, $C_2 = 8/9$, $C_3 = -1/3$. Hence, the unique solution of the recursion is

$$a_n = -\frac{1}{9} \cdot (-1)^n + \left(\frac{8}{9} - \frac{n}{3}\right) \cdot 2^n.$$

**5. (i)** $a_n = n$, which can be verified by

STEP 1: Check the initial conditions: yes, the formula works for $n = 0, 1$.

STEP 2: Check that the formula satisfies the recursion: yes, since $n = 2(n-1) - (n-2)$.

OBS! Formally, what you're doing here is proving the formula $a_n = n$ by so-called *strong induction* on $n$.

**(ii)** The characteristic equation is $\alpha^2 = 2\alpha - 1$, which has the repeated root $\alpha_{1,2} = 1$. Hence the general solution is

$$q_n = C_1 \cdot 1^n + C_2 \cdot n \cdot 1^n = C_1 + C_2 \cdot n.$$

Insert the initial conditions:

$$n = 0: \quad a_0 = 0 = C_1,$$
$$n = 1: \quad a_1 = 1 = C_1 + C_2 \Rightarrow C_1 = 1.$$

Hence, $a_n = n$, v.s.v.