Third Exercise Session: 17/4

Theme: Inhomogenous linear recursions, Generating functions, Non-linear recursions

Relevant Chapters: Vol. 2: 4.2.2, 6.4, 6.6

1. Redo Example 4.5 in the lecture notes using instead the method of generating functions, i.e.: solve the recursion

$$u_0 = u_1 = 1, \quad u_n = 6u_{n-1} - 5u_{n-2} + 5^n + n + 1 \; \forall \, n \geq 2.$$

2. (6.6.19 in Vol. 2) Let $A(x)$ and $B(x)$ be the generating functions of the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ respectively.

(a) For which sequence is $A(x) + B(x)$ the generating function ?
(b) For which sequence is $A(x)B(x)$ the generating function ?
(c) For which sequence is $A(x^2)$ the generating function ?
(d) For which sequence is $A'(x)$ the generating function ?
(e) For which sequence is $(A(x) - a_0)/x$ the generating function ?
(f) Let a_{-1} be some number. For which sequence is $xA(x)+a_{-1}$ the generating function ?

3. A permutation $\pi_1\pi_2 \ldots \pi_n$ of the numbers $1,\, 2,\, \ldots,\, n$ is said to be *1-3-2 avoiding* if there does not exist any triple $(i,\, j,\, k)$ such that $i < j < k$ and $\pi_i < \pi_j > \pi_k > \pi_i$. Let A_n be the number of 1-3-2 avoiding permutations of $1,\, 2,\, \ldots,\, n$.
(a) Compute $A_1,\, A_2,\, A_3,\, A_4$ directly by writing down all possible permutations.
(b) Prove that $A_n = C_n$ for every $n \in \mathbb{N}$, where C_n is the n:the Catalan number.
(HINT: Show that the numbers A_n satisfy the same recursion as the numbers C_n).

4. Recall that d_n denotes the number of derangements of $1,\, 2,\, \ldots,\, n$, i.e.: the number of permutations $\pi_1\pi_2 \ldots \pi_n$ such that $\pi_i \neq i$ for every i. Prove that, for all $n \geq 2$,

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

Solutions

1. Set $G(x) = \sum_{n=0}^{\infty} u_n x^n$.

Step 1: Express $G(x)$ as a rational function.

We begin as in Example 5.5 and obtain

$$\begin{aligned}(1-6x+5x^2)G(x) &= (u_0+u_1x) - 6u_1x + \sum_{n=2}^{\infty}(u_n - 6u_{n-1} + 5u_{n-2})x^n \\ &= (1+x) - 6x + \sum_{n=2}^{\infty}(5^n + n + 1)x^n \\ &= 1 - 5x + \sum_{n=2}^{\infty} 5^n x^n + \sum_{n=2}^{\infty}(n+1)x^n. \qquad (1)\end{aligned}$$

Each of the two sums needs to be expressed as a rational function, so we take them in turn. The first is just a geometric series:

$$\sum_{n=2}^{\infty} 5^n x^n = \sum_{n=2}^{\infty}(5x)^n = \frac{(5x)^2}{1-5x} = \frac{25x^2}{1-5x}.$$

For the second sum we compute as follows:

$$\begin{aligned}\sum_{n=2}^{\infty}(n+1)x^n &= \sum_{n=0}^{\infty}(n+1)x^n - (1+2x) \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) - (1+2x) \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) - (1+2x) \\ &= \frac{1}{(1-x)^2} - (1+2x).\end{aligned}$$

Substituting everything into (1) gives

$$\begin{aligned}(1-6x+5x^2)G(x) = 1 - 5x + \frac{25x^2}{1-5x} + \frac{1}{(1-x)^2} - (1+2x) \Rightarrow \\ (1-5x)(1-x)G(x) = \frac{-7x(1-5x)(1-x)^2 + 25x^2(1-x)^2 + (1-5x)}{(1-5x)(1-x)^2} \\ \Rightarrow \cdots \Rightarrow G(x) = \frac{60x^4 - 127x^3 + 74x^2 - 12x + 1}{(1-5x)^2(1-x)^3}.\end{aligned}$$

Step 2: The partial fraction decomposition looks like

$$\frac{60x^4 - 127x^3 + 74x^2 - 12x + 1}{(1-5x)^2(1-x)^3} = \frac{A}{1-5x} + \frac{B}{(1-5x)^2} + \frac{C}{1-x} + \frac{D}{(1-x)^2} + \frac{E}{(1-x)^3}.$$

After multiplying up by the common denominator and comparing coefficients of each power of x, we'll be left with a system of 5 linear equations for the 5 unknowns

A, B, C, D, E. I used `Wolfram Alpha` to perform the computation and got

$$A = -\frac{167}{64}, \quad B = \frac{5}{4}, \quad C = \frac{187}{64}, \quad D = -\frac{5}{16}, \quad E = -\frac{1}{4}. \tag{2}$$

Step 3: Use Theorem 5.1 to convert the partial fraction decomposition back into a power series:

$$G(x) = A\left(\sum_{n=0}^{\infty}(5x)^n\right) + B\left(\sum_{n=0}^{\infty}(n+1)(5x)^n\right)$$
$$+C\left(\sum_{n=0}^{\infty}x^n\right) + D\left(\sum_{n=0}^{\infty}(n+1)x^n\right) + E\left(\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n\right).$$

Comparing coefficients of x^n we deduce that

$$\begin{aligned} u_n &= A\cdot 5^n + B\cdot(n+1)\cdot 5^n + C + D(n+1) + E\,\frac{(n+1)(n+2)}{2} \\ &= ((A+B)+Bn)\cdot 5^n + \frac{E}{2}n^2 + \left(D + \frac{3E}{2}\right)n + (C+D+E). \end{aligned}$$

Finally, inserting the values from (2) yields

$$u_n = \left(-\frac{87}{64} + \frac{5n}{4}\right)\cdot 5^n - \frac{n^2}{8} - \frac{11n}{16} + \frac{151}{64}.$$

2. Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ denote the power series of interest in each part of the exercise. We must therefore express the c_n in terms of the a_n and the b_n.

(a)

$$C(x) = A(x) + B(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty}(a_n + b_n)x^n,$$

so $c_n = a_n + b_n$ for every n.

(b)

$$C(x) = A(x)B(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} x^n \left(\sum_{m=0}^{n} a_m\, a_{n-m}\right),$$

so $c_n = \sum_{m=0}^{n} a_m a_{n-m}$.

(c) $A(x^2) = \sum_{n=0}^{\infty} a_n (x^2)^n = \sum_{n=0}^{\infty} a_n x^{2n}$, so

$$c_n = \begin{cases} a_{n/2}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(d) $A'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$, so $c_n = (n+1)a_{n+1}$.

(e)

$$\frac{A(x) - a_0}{x} = \frac{1}{x}\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} x^n,$$

so $c_n = a_{n+1}$.

(f)

$$xA(x) - a_{-1} = \sum_{n=0}^{\infty} a_n x^{n+1} - a_{0-1}x^0 = \sum_{n=1}^{\infty} a_{n-1}x^n - a_{0-1}x^0 = \sum_{n=0}^{\infty} a_{n-1}x^n,$$

so $c_n = a_{n-1}$.

3. (a) $A_1 = 1! = 1$ and $A_2 = 2! = 2$ since a permutation on fewer than three numbers cannot contain any pattern involving three numbers. For $n = 3$, the only permutation on three numbers which has the 1-3-2 pattern is the permutation 132 itself, so $A_3 = 3! - 1 = 5$. For $n = 4$, it turns out that $A_4 = 14$ and the $4! - 14 = 10$ permutations with the 1-3-2 pattern are

$$1243,\ \ 1324,\ \ 1342,\ \ 1423,\ \ 1432,\ \ 2143,\ \ 2413,\ \ 2431,\ \ 3142,\ \ 4132.$$

(b) Note that also $A_0 = 1$ since the only permutation of the empty set is the empty permutation, which contains no patterns at all. Since thus $A_0 = C_0 = 1$, it suffices to show that, for every $n \geq 1$,

$$A_n = \sum_{m=1}^{n} A_{m-1}A_{n-m}. \tag{3}$$

Consider those 1-3-2 avoiding permutations of $1, 2, \ldots, n$ where n is placed in the m:th position. If any number placed to the left of n were smaller than any number placed to the right of it, then these two together with n would form a 1-3-2 pattern.

Hence, the $m - 1$ numbers to the left of n must form a permutation of $n - m + 1, \ldots, n - 1$, which is therefore just a shift of a permutation of $1, 2, \ldots, m - 1$. This permutation must itself avoid the 1-3-2 pattern and so there are A_{m-1} choices for it.

Similarly, the $n-m$ numbers to the right of n must form a 1-3-2 avoiding permutation of $1, 2, \ldots, n - m$ so there are A_{n-m} choices for it.

Finally, we note that the entire permutation of $1, 2, \ldots, n$ will avoid the 1-3-2 pattern if and only if those parts of it on either side of n do so. Hence, given that n is placed in position m there are, by MP, $A_{m-1}A_{n-m}$ choices for the entire permutation. Since m can run from 1 through to n, summing over m proves (3).

4. There are $n - 1$ choices for the position of 1 in a derangement of $1, 2, \ldots, n$ and clearly the number of possible derangements is independent of where we put 1. So $d_n = (n - 1)T$, where T is the number of derangements where 1 is placed in position 2, say.

Case 1: 2 is placed in position 1. Then it remains to make a derangement of the $n-2$ numbers $3, 4, \ldots, n$ and so there are d_{n-2} possibilities.

Case 2: 2 is not placed in position 1. Here the idea is to "identify" positions 1 and 2 and thus imagine that one is left having to make a derangement of $2, 3, \ldots, n$. There are thus d_{n-1} possibilities in this case.

Thus, by AP, $T = d_{n-2} + d_{n-1}$ and so $d_n = (n - 1)(d_{n-2} + d_{n-1})$, v.s.v.