Seventh Exercise Session: 11/5

Theme: Graph theory

Relevant Chapters: Vol.1: 6

1. A finite sequence of positive integers is said to be *graphic* if there exists a simple, loopless graph with these numbers as the degrees of its vertices. Which of the following sequences are graphic ? Motivate your answers !

$3, 3, 3, 3, 3, 3, 3$
$3, 3, 3, 3, 3, 3, 4$
$1, 2, 2, 2, 3, 6, 6$
$2, 2, 2, 2, 2, 6, 6$
$2, 2, 2, 3, 4, 4, 5.$

2. Graph theorist Anon Ymous went to Disneyworld and visited the House of Horrors, see Figure D7.2. For the fun of it, he wondered if he could visit all seven rooms and the surrounding corridor without passing through any door more than once. Determine if and how Mr Ymous can do this

(i) if he needs to end up back in the corridor so he can get home before the place closes

(ii) if, just to be extra naughty, he decides he'd like to hide out in one of the rooms overnight.

3. For each of the graphs in Figure D7.3, determine whether or not it has a Hamilton path and/or cycle. Motivate your answers !

4. For each $n \in \mathbb{N}$, let $Q_n = (V_n, E_n)$ denote the n-dimensional binary hypercube, i.e.: V_n is the set of all n-bit binary strings and we place an edge between two strings if and only if they differ in exactly one bit.

(i) Draw Q_1, Q_2 and Q_3.

(ii) For which $n \in \mathbb{N}$ does Q_n possess a Hamilton path resp. cycle ?

(iii) Determine $\chi(Q_n)$ for every n.

Solutions

1. The first sequence is not graphic since the sum of the degrees is odd (Theorem 15.4). The third sequence is not graphic since the two sixes imply there are two vertices joined to all others, which means every vertex would have to have degree at least two. The other three sequences are graphic - see Figure D7.1(S) for examples.

2. We construct a graph $G = (V, E)$ whose vertices are the eight "rooms" (i.e.: the seven actual rooms plus the corridor), and where each edge represents a door between adjacent rooms. See Figure D7.2(S), in which the rooms are numbered left-to-right and then top-to-bottom, the corridor being vertex nr. 8. All eight vertices have even degree. This means that Mr Ymous can achieve his goal and get back out to the corridor and home in time to avoid arrest. An example (there are many) of an Euler circuit starting and ending in the corridor is

$$8 \to 4 \to 3 \to 7 \to 3 \to 2 \to 6 \to 5 \to 1 \to 8 \to 7 \to 6 \to 8 \to 5 \to 8.$$

Note that there are doors to the corridor in rooms 1, 4, 5, 6 and 7. If we removed exactly one of these doors (i.e.: filled in the wall where the door was), then there would be an Euler trail from the corridor to that room and Mr. Ymous could hide out there overnight.

3. (a) There exist Hamiltonian cycles, e.g.:

$$a \to g \to k \to i \to h \to b \to c \to d \to j \to f \to e \to a.$$

(b) There exist Hamiltonian cycles, e.g.:

$$a \to d \to b \to e \to g \to j \to i \to f \to h \to c \to a.$$

(c) There exist Hamiltonian cycles, e.g.:

$$a \to b \to e \to f \to g \to c \to d \to i \to h \to a.$$

(d) There exist Hamiltonian paths, e.g.:

$$a \to c \to e \to b \to d \to f \to g.$$

However, there are no Hamiltonian cycles. Each of a, d, g is connected to e and one other vertex. In a Hamilton cycle we must enter and leave every vertex exactly once, along different edges. This means that we cannot avoid hittingthe vertex e three times, which is not allowed since it can only be visited once.

(e), (f) There exist Hamiltonian paths in the 3×5 grid, e.g.:

$$a \to b \to c \to d \to e \to j \to i \to$$
$$\to h \to g \to f \to k \to l \to m \to n \to o.$$

There exist Hamiltonian cycles in the 4×5 grid, e.g.:

$$a \to b \to c \to d \to e \to j \to i \to h \to g \to l \to m \to$$
$$\to\to n \to o \to t \to s \to r \to q \to p \to k \to f \to a.$$

The general theorem here is the following:

Theorem. *In an $m \times n$ rectangular grid there is always a Hamiltonian path, but there is a Hamiltonian cycle if and only if mn is even and $\min\{m, n\} \geq 2$.*

PROOF: It's easy to see that there is always a Hamiltonian path starting at the top left-hand corner and ending at the bottom right-hand corner, namely just zig-zag your way back and forth along the rows. Similarly, if the number of rows is even and there are at least two of them, then, starting from the top left-hand corner, we can zig zag along the rows as before, but this time avoiding the lefthand column in rows 2 to $m-1$. We will arrive at the bottom right-hand corner, and can then go leftwards along the bottom row and back up along the leftmost column to obtain a Hamiltonian cycle (in the form of an E-shaped figure). If instead the number of columns is even and greater than one, we can perform the same procedure except we interchange the role of rows and columns (imagine rotating the grid 90 degrees; the Hamilton cycle will now be an M-shaped figure). This proves that a Hamiltonian cycle exists provided mn is even and $\min\{m, n\} \geq 2$. It is also obvious that no Hamiltonian cycle exists if there is just one row or just one column. So it remains to prove there is no Hamiltonian cycle if mn is odd.

Let $G = G_{m,n} = (V, E)$ denote the $m \times n$ grid graph. The crucial point is that this graph is bipartite. Namely, there is a bipartition $V = (V_1, V_2)$ such that V_1 (resp. V_2) contains all vertices in positions (i, j) such that $i + j$ is even (resp. $i + j$ is odd). There are $\lceil \frac{mn}{2} \rceil$ vertices in V_1 and $\lfloor \frac{mn}{2} \rfloor$ vertices in V_2. Hence, V_1 and V_2 contain exactly the same number of vertices if and only if mn is even. But in any bipartite graph $G = (V_1, V_2, E)$, a necessary condition for existence of a Hamiltonian cycle is that $|V_1| = |V_2|$, for any path in the graph must go back and forth between V_1 and V_2.

4. (i) See Figure D7.4(S).

(ii) Q_1 consists of a single edge, hence possesses a Hamilton path but not a Hamilton cycle. I claim that Q_n possesses a Hamilton cycle for every $n \geq 2$. The standard way to prove this is by induction on n.

BASE CASE: An example of a Hamilton cycle in Q_2 is

$$(0, 0) \to (1, 0) \to (1, 1) \to (0, 1) \to (0, 0).$$

INDUCTION STEP: Let $n \geq 2$ and suppose Q_n possesses a Hamilton cycle, starting and ending at the all-zeroes node $\mathbf{0}_n$, say

$$\mathbf{0}_n = \boldsymbol{v}_1 \to \boldsymbol{v}_2 \cdots \to \boldsymbol{v}_{2^n} \to \mathbf{0}_n.$$

For each $i = 1, \ldots, 2^n$ and $j \in \{0, 1\}$, let $\boldsymbol{w}_{j,i} \in Q_{n+1}$ be the string whose first bit is j and whose remaining bits form $\boldsymbol{v}_i \in Q_n$. A Hamilton cycle in Q_{n+1} starting and ending at $\mathbf{0}_{n+1} = \boldsymbol{w}_{0,1}$ is then given by

$$\boldsymbol{w}_{0,1} \to \boldsymbol{w}_{0,2} \to \cdots \to \boldsymbol{w}_{0,2^n} \to \boldsymbol{w}_{1,2^n} \to \cdots \to \boldsymbol{w}_{1,2} \to \boldsymbol{w}_{1,1} \to \boldsymbol{w}_{0,1}.$$

(iii) Every vertex $\boldsymbol{v}$ of Q_n is a string of n ones and zeroes. Let $s(\boldsymbol{v}) = 1$ if the sum of the bits is odd and $s(\boldsymbol{v}) = 0$ if the sum of the bits is even. Let V_o (resp. V_e) be the subset of those vertices $\boldsymbol{v} \in Q_n$ such that $s(\boldsymbol{v}) = 1$ (resp. $s(\boldsymbol{v}) = 0$). Then clearly every edge in Q_n is between a vertex in V_o and a vertex in V_e. In other words, Q_n is a bipartite graph for every n and hence $\chi(Q_n) = 2$ for every n.