Eighth Exercise Session: 17/5

Theme: Graph theory

Relevant Chapters: Vol.1: 6; Vol. 2: 7.1, 7.2, 7.4, 8.2

1. Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $\phi : V_1 \to V_2$ such that $\{v_1, w_1\} \in E_1 \Leftrightarrow \{\phi(v_1), \phi(w_1)\} \in E_2$.

Determine which pairs of graphs in Figure D8.1 are isomorphic. Motivate your answers !

2. For the graph G in Figure D8.2,

(i) Give at least four different minimally non-3-colorable subgraphs of G, i.e.: subgraphs H s.t. $\chi(H) > 3$ butr $\chi(H') \leq 3$ for every proper subgraph of H.

(ii) Determine $\chi(G)$.

(iii) Give an ordering of the vertices for which the greedy algorithm does not produce an optimal coloring.

3. Consider the weighted, undirected graph in Figure D8.3.

(i) Apply Prim's algorithm to find a MST in G, starting from the vertex s. Indicate clearly which edge you choose at each step and the total weight of the final tree.

(ii) Apply instead Kruskal's algorithm to find a MST. Again, indicate clearly which edge you choose at each step and the total weight of the final tree.

(iii) How many different MSTs are there in G ?

(iv) Apply Dijkstra's algorithm to find a shortest path from s to t. Indicate clearly which edge is chosen and which label is made permanent at each step, along with the final path and its total length. (OBS! Since G is undirected, you may go in either direction along any edge.)

4. (i) Prove that if G is a plane graph with at least three vertices then $e \leq 3v - 6$.
(ii) Deduce that $\chi(G) \leq 6$ for any plane graph G.

Solutions

1. The first pair are isomorphic, and there are two possible isomorphisms, see Figure D8.1(S). The second pair are not - for example, since the graph on the left contains cycles of length 3 whereas that on the right does not.

2. (i) Three such configurations are formed by the induced subgraphs on $\{A, B, D, E\}$, $\{E, F, G, I\}$ and $\{C, E, H, I\}$, each of which is a K_4. A fourth configuration is the wheel graph W_5 centered at E, with spokes pointing to A, B, G, I, C.

(ii) $\chi(G) = 4$. One way to see this is that G is planar - see Figure D8.2(S) for a plane redrawing of G. An example of an explicit 4-coloring $\chi : V(G) \to \{1, 2, 3, 4\}$ is given by

$$\chi(A) = \chi(G) = \chi(H) = 1, \quad \chi(B) = \chi(C) = \chi(F) = 2, \quad \chi(I) = 3, \quad \chi(E) = 4.$$

(iii) If we order the vertices in alphabetical order then the colors assigned by the greedy algorithm will be, in order, $1, 2, 2, 3, 4, 1, 3, 1, 5$.

3. (i) See table.

Step	Edge chosen	Weight
1	$\{s, a\}$	2
2	$\{s, d\}$	3
3	$\{d, c\}$	1
4	$\{c, f\}$	3
5	$\{f, g\}$	1
6	$\{g, j\}$	2
7	$\{j, i\}$	3
8	$\{i, l\}$	2
9	$\{l, t\}$	3
10	$\{d, e\}$	3
11	$\{b, e\}$	2
12	$\{e, h\}$	3
13	$\{t, m\}$	4
14	$\{k, m\}$	1
Total weight		33

(ii) See table.

(iii) Observe that we get the same tree in parts **(i)** and **(ii)** and that this tree contains every edge in G of weight 1, 2 or 3, plus a single edge of weight 4. Hence, any MST must have the same property. But once all edges of weight at most 3 are present, then we can't include $\{a, c\}$ or $\{d, g\}$, because either would create a cycle. So the only choice left for a weight-4 edge is $\{t, m\}$. Hence the MST is unique.

Step	Edge chosen	Weight
1	$\{c,\, d\}$	1
2	$\{f,\, g\}$	1
3	$\{k,\, m\}$	1
4	$\{s,\, a\}$	2
5	$\{b,\, e\}$	2
6	$\{g,\, j\}$	2
7	$\{i,\, l\}$	2
8	$\{s,\, d\}$	3
9	$\{c,\, f\}$	3
10	$\{d,\, e\}$	3
11	$\{e,\, h\}$	3
12	$\{i,\, j\}$	3
13	$\{l,\, t\}$	3
14	$\{t,\, m\}$	4
Total weight		33

(iv)

Step	Edge added	Label added
1	$\{s,\, a\}$	$l(a) := 2$
2	$\{s,\, d\}$	$l(d) := 3$
3	$\{d,\, c\}$	$l(c) := 4$
4	$\{s,\, b\}$	$l(b) := 5$
5	$\{d,\, e\}$	$l(e) := 6$
6	$\{c,\, f\}$	$l(f) := 7$
7	$\{d,\, g\}$	$l(g) := 7$
8	$\{g,\, j\}$	$l(j) := 9$
9	$\{e,\, h\}$	$l(h) := 9$
10	$\{j,\, i\}$	$l(i) := 12$
11	$\{i,\, l\}$	$l(l) := 14$
12	$\{h,\, k\}$	$l(k) := 14$
13	$\{j,\, t\}$	$l(t) := 15$

Note that steps 6-7, 8-9 and 11-12 are interchangeable. The unique $s \leftrightarrow t$ path in this tree is found by reading backwards from t:

$$s \leftrightarrow d \leftrightarrow g \leftrightarrow j \leftrightarrow t.$$

4. (i) We apply Euler's theorem in the form $v - e + r = 2$, thus counting the exterior of the graph as a region. Consider the set of pairs

$$\mathcal{S} = \{(\varepsilon,\, \rho) : \varepsilon \text{ is an edge on the boundary of region } \rho.$$

On the one hand, each edge is either on the boundary between exactly two regions or is a "hanging edge" which only bounds the exterior region. Hence $|\mathcal{S}| \leq 2e$. On the other hand, since the graph is simple, each region must be bounded by at least three edges, so $|\mathcal{S}| \geq 3r$.

It follows that $3r \leq 2e$, thus $r \leq 2e/3$. Substituting into Euler's formula we get

$$e = v + r - 2 \leq v - 2 + \frac{2e}{3} \Rightarrow \frac{e}{3} \leq v - 2 \Rightarrow e \leq 3v - 6, \text{ v.s.v.}$$

(ii) Induction on $v = |V(G)|$. Clearly the result holds if $v < 3$, since trivially $\chi(G) \leq |V(G)|$. Suppose the result holds for all plane graphs on at most $n \geq 3$ vertces and let G be a plane graph with $n+1$ vertices. By part **(i)** and the degree equation (Theorem 15.4) it follows that G must possess a vertex of degree at most 5. Let v_0 be any such vertex and let G' be the graph obtained by removing v_0 and all its adjacent edges. This graph is still plane and has n vertices, so by the induction assumption it can be colored with at most 6 colors. But since v_0 has degree at most 5 in G, at least one of the 6 colors is not used on any neighbor of v_0 and hence the 6-coloring of G' can be extended to a 6-coloring of G, v.s.v.