Ninth Exercise Session: 25/5

Theme: Graph theory

Relevant Chapters: Vol. 2: 8.3, 9.1, 9.2

1. (i) Starting from the empty matching $M = \phi$, apply the augmenting path algorithm to find a maximum matching in the graph in Figure D9.1. Write down the augmenting path located and the new matching obtained at each step.

(ii, see Övning 9.6) In a graph $G = (V, E)$, the *deficiency* d_A of a set $A \subseteq V$ is given by $d_A := \max\{0, |A| - |N(A)|\}$. For the graph in Figure D9.1, give examples of subsets of X and Y of maximum deficiency. What do you notice ?

2. (Övning 9.30) Kimmo har bjudit in Olle, Ambjörn, Lasse och Viggo på fruktsallad. Sammanlagt behövs alltså fem portioner. Kimmo låter fantasin flyga och komponerar följande portioner:

- äpple, banan, apelsin, melon och nötter
- banan, apelsin och russin
- banan, apelsin, melon, nötter och russin
- banan, russin, melon och nötter
- äpple, banan, apelsin och nötter.

När gästerna kommer visar det sig dock att Kimmo inte gillar nötter, att Lasse inte tål apelsin och att Viggo är alergisk mot äpple, medan Amnbjörn absolut vill ha en portion med båda banan och russin i. Olle äter dock vilken fruktsallad som helst.

(a) Dela ut så många lämpliga portioner som möjligt, det vill säga rita en lämplig bipartit graf och hitta en maximum matchning.
(b) Avgör om den bipartita graf du fick är planär eller inte.

3. Apply the Ford-Fulkerson algorithm to find a maximum flow and minimum cut in the network in Figure D9.3. Write down the augmenting path chosen and the increase in flow strength at each step. Draw the final flow in full.

Solutions

1. (i) There are many alternatives, below is one. Squiggly arrows denote edges in the current matching.

Step	Augmenting path	New matching
1	$1 \to B$	$\{1,\, B\}$
2	$D \to 1 \rightsquigarrow B \to 4$	$\{1,\, D\},\ \{4,\, B\}$
3	$3 \to C$	$\{1,\, D\},\ \{3,\, C\},\ \{4,\, B\}$
4	$5 \to F$	$\{1,\, D\},\ \{3,\, C\},\ \{4,\, B\},\ \{5,\, F\}$
5	$2 \to F \rightsquigarrow 5 \to A$	$\{1,\, D\},\ \{2,\, F\},\ \{3,\, C\},\ \{4,\, B\},\ \{5,\, A\}$

(ii) We have $d_{\mathcal{A}} = d_{\mathcal{B}} = 1$ where

$$\mathcal{A} = \{1,\, 4,\, 6\} \;\Rightarrow\; N(\mathcal{A}) = \{B,\, D\}, \quad \mathcal{B} = \{A,\, C,\, E,\, F\} \;\Rightarrow\; N(\mathcal{B}) = \{2,\, 3,\, 5\}.$$

This is consistent with the following extension of Hall's Theorem. For a bipartite graph $G = (X,\, Y,\, E)$, define the deficiency of X by

$$\delta_X := \max_{A \subseteq X} d_A.$$

Extended Hall's Theorem. *The maximum size of a matching in a bipartite graph $G = (X,\, Y,\, E)$ is $|X| - \delta_X$ (which is the same as $|Y| - \delta_Y$).*

PROOF: This is a generalisation of Theorem 19.8 (the case $\delta_X = 0$).

Firstly, let A be a subset of X of maximum deficiency, i.e.: $d_A = \delta_X$. Then, since elements of A can only be matched with elements of $N(A)$, at least $d_A = \delta_X$ elements of A, and hence of X, must be left unmatched in any matching. This proves that there is no matching of size greater than $|X| - \delta_X$.

It remains to prove there exists a matching of size $|X| - \delta_X$. Define a new bipartite graph $G^* = (X,\, Y^*,\, E^*)$, where

- $Y^* = Y \sqcup Y_1$, where $|Y_1| = \delta_X$
- $E^* = E \sqcup E_1$, where E_1 consists of an edge from every element of X to every element of Y_1.

By construction, the deficiency of X is zero in G^*. Hence, by Theorem 19.8, G^* possesses an X-perfect matching. But in such a matching, at most δ_X of the vertices of X can be matched with a vertex in Y_1, hence at least $|X| - \delta_X$ of them are matched with vertices in Y. This constitutes a matching in the original G of size at least $|X| - \delta_X$, v.s.v.

2. (a) We define a bipartite graph $G = (X,\, Y,\, E)$ such that

- X is the set of people, i.e.: $X = \{K,\, O,\, A,\, L,\, V\}$,
- Y is the set of possible portions, i.e.: $Y = \{P1,\, P2,\, P3,\, P4,\, P5\}$
- We place an edge whenever the respective portion is acceptable for the respective guest.

The resulting G is illustrated in Figure D9.2(S). There cannot be a perfect matching since, for example, if $\mathcal{A} = \{O,\, A,\, L,\, V\}$ then $N(\mathcal{A}) = \{P2,\, P3,\, P4\}$ so $d_{\mathcal{A}} = 1$. We can, however, find a matching of size $5 - 1 = 4$, for example

$$M = \{\{K,\, P2\},\, \{O,\, P1\},\, \{A,\, P3\},\, \{L,\, P4\}\}.$$

(b) G is not planar, because it contains a $K_{3,3}$-subgraph formed by $\{O, A, V\}$ and $\{P2, P3, P4\}$.

3. An example of how the algorithm might proceed is as follows (as in Q.1, there are many alternatives):

Step	f-augmenting path	Increase in flow strength
1	$s \to a \to d \to g \to t$	8
2	$s \to b \to e \to h \to t$	6
3	$s \to c \to f \to j \to t$	5
4	$s \to b \to e \to d \to g \to h \to t$	2
5	$s \to b \to e \to g \to h \to t$	2
6	$s \to c \to e \to g \to t$	1
7	$s \to c \to e \to f \to h \to t$	4
8	$s \to c \to e \to f \to i \to h \to t$	3
Total flow strength		31

The resulting maximum flow is illustrated in Figure D9.3(S). The set of nodes reachable from s by an augmenting path is then $S = \{s, a, b, c, d, e, f\}$. Set $T = V \backslash S = \{g, h, i, j, t\}$. Then

$$c(S, T) = c(d, g)+c(e, g)+c(e, h)+c(f, h)+c(f, i)+c(f, j) = 10+3+6+4+3+5 = 31 = |f|.$$